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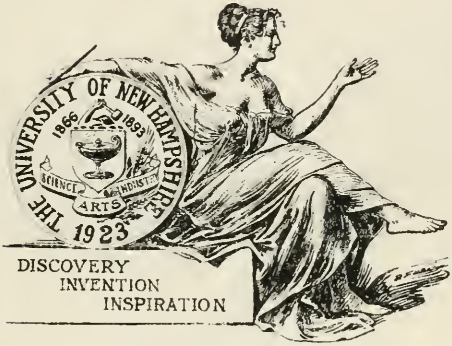

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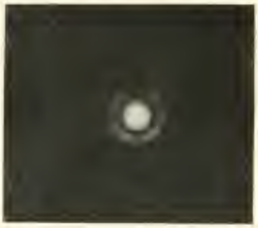
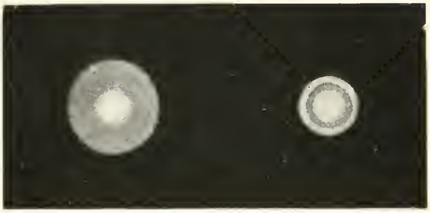
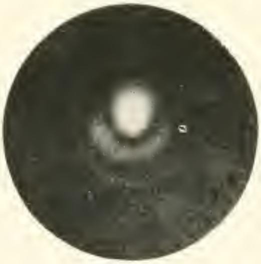


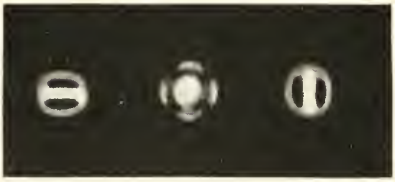
(*a*)
The "Airy" Disc



(*b*)
Effects near the focus in the presence of spherical aberration



(*c*)
The effect of a small amount of coma



(*d*)
The effects, near the focus, of a small amount of astigmatism

"STAR" IMAGES

[*Frontispiece*

OPTICAL MEASURING INSTRUMENTS

Their Construction, Theory, and Use

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BY

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FOREWORD

By Professor F. J. CHESHIRE, C.B.E.

Director of the Optical Engineering and Applied Optics Department of the Imperial College
of Science and Technology, South Kensington

I accede with pleasure to the request that I should write a Foreword for the book of my colleague, Dr. Martin, on *Optical Measuring Instruments*. More especially do I do so, because it was at my invitation that he originally turned his attention to the subject, when the Department of Optical Engineering and Applied Optics was founded at the Imperial College.

In Dr. Martin's book the student will find, brought together for the first time, descriptions of a number of well-selected typical instruments, which embody and exemplify in their design, construction, and use, optical principles and devices of great interest and of wide application—information which hitherto it has been very difficult if not impossible to obtain except by reference to the records of the Patent Office, or those of learned societies. This bringing together of cognate matter should be of great value and convenience to everyone interested in the application of optical measuring means to instruments of precision generally.

The limitations, for measuring purposes, of the human eye were humorously brought out by a friend of mine—a famous inventor of range-finders—when, some years ago, he deplored the fact that nature had not seen fit to impress our retinas with stereoscopic scales, by means of which ranges could be accurately determined without any instrumental aid. It is curious to notice, bearing this view in mind, how often, as set out by Dr. Martin, a measuring instrument does, in effect, resolve itself into an artificial eye of enhanced power, with a scale imprinted on its retina, mechanically

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mounted so that its movements in various directions can be measured with precision—the use of the natural eye being restricted to the observation of the retina of the artificial one, either directly or with the aid of a simple magnifier.

Dr. Martin's book appeals then not only to the student and optician, but to everyone interested in the adoption of methods and instruments of greater and greater precision in the Arts and Sciences.

F. J. CHESHIRE.

PREFACE

Up till the beginning of the nineteenth century the study and development of instruments held a position of the very greatest importance in the scientific world, and the contributions of instrument makers occupied a large proportion of the pages of the *Philosophical Transactions of the Royal Society*. *Autre temps*, *autres mœurs!* It seems that the tendency of the schools has been to concentrate increasingly on the philosophic aspects of science, and the graduate who strays by chance into the workshops and laboratories of a modern instrument maker is frequently amazed to find a standard of scientific practice in some respects in advance of his previous experience. It may be noted in passing that the practice of many of our British firms is second to none.

The present book makes no pretence of instructing the optical scientific instrument maker in his own work, but since it presents some of the views of the user of instruments, and since it covers a range of instruments wider than the production of most factories, it may prove of some interest to commercial firms.

No user of a tool can study it too thoroughly, and he will learn a lesson of no little value if he realizes that, in the use of an optical instrument, the eye and, in fact, the whole mind and body are equally concerned in obtaining the required results.

The author will feel amply repaid, however, if it should prove that the book appeals to teachers of science. Are there not many parts of typical physics courses which could with advantage be replaced by more modern material which, while equally educational, is more likely to find direct application in practice? The study of instruments yields such material in abundance.

An effort has been made to avoid the introduction of matter which is available in the usual text-books. The selection of instruments for description is, however, extremely difficult, and it is not to be concluded that those which are described are the only good ones of their class, although an attempt has been made to give an account of the best features of modern practice irrespective of any other consideration. If undue prominence seems to be given here and there to individual work and productions, it must be remembered that a writer is necessarily influenced most by those with whom he is most in contact. Any person choosing an instrument will do well to study the productions of all the firms which make such a one.

I am greatly indebted to many friends for valuable help in the preparation of the book, and especially to Professor Cheshire, C.B.E., who contributes a foreword. To Mr. L. Bellingham, of Messrs. Bellingham & Stanley, Ltd.; Professor A. E. Conrady, A.R.C.S.; Major E. O. Henrici, R.E.; Mr. C. H. Kebby, A.R.C.S., B.Sc.; Mr. Cecil Mason, of the Cambridge & Paul Instrument Co., Ltd.; and Mr. G. W. Watts, of Messrs. E. R. Watts & Son, Ltd., I have to return grateful thanks for their kind advice. I am greatly indebted to the firms mentioned above, which have kindly supplied material for illustrations of several instruments, as have also Messrs. Cooke, Troughton, & Simms, Ltd., Adam Hilger, Ltd., W. Wilson & Co., Ltd., the Société Génevoise, and Carl Zeiss (through Messrs. Atha, Ltd.). Messrs. The J. B. Lippincott Company have kindly granted permission to copy a drawing (fig. 42) of a transit instrument from Chauvenet's classical work *Spherical and Practical Astronomy*, which has been of great value in reading for some parts of the present book. Chauvenet's notation has been adopted in the elementary discussion of the divided circle in Chapter IV, so that those who wish to pursue the subject can refer to the fuller original treatment. Such papers as Mr. R. S. Whipple's admirable address on "The Design and Construction of Scientific Instruments" in the *Transactions of the Optical Society* (Vol. XXII) have also proved very useful, and mention must be made of the recent valuable papers due to the staff of the National Physical Laboratory. The remaining sources of the material are too numerous to mention in detail, but frequent references will be found in the text.

L. C. MARTIN.

South Kensington,
1st May, 1924

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OPTICAL MEASURING INSTRUMENTS

CHAPTER I

General Principles

The Instrument as a Tool.—The object of the present work is to furnish a short account of the more important optical measuring instruments, and the theory underlying their construction and use. It may be instructive to begin with a few general considerations which will assist in making an orderly approach to the subject.

The dictionary defines an instrument as " the means whereby something is effected ": it will be recognized then that in general we are to regard instruments as *tools* devoted to certain specific uses. The construction of the instrument should therefore relate always to the conditions of use, and to the convenience of the user. In these connections both *design* and *materials* are important, and in considering the materials for the construction of any tool the place of use and climatic conditions are among the numerous points to be remembered.

Evolution of Instruments. — Many instruments have " grown ", or rather have been evolved during the course of time. Various parts have been added or grouped together; perhaps the instrument as a whole has been applied to different uses for which the original design is unsuitable. It is therefore necessary always to make an effort to consider the construction while bearing in mind the *present* need which the tool is meant to supply.

The modern theodolite may be cited as a typical example of evolution. The ancient altazimuth instrument with its divided circles and pointers, combined with the modern telescope evolved from simple lenses and their supports, forms the modern theodolite. Not one brain, but many, has contributed to the making of most of our complex instruments.

In former times refractive indices were measured with a prism and theodolite; the realization that the measurement could be greatly simplified by the addition of a collimator, and the displacement of the telescope, led at once to the design of the spectrometer.

Human Observation.—As a generalization it may be stated that nearly all human observation consists in the recognition of coincidences and equalities. Estimation enters to some extent into scientific work, as, for example, in estimating the position of a fiducial mark between two divisions on a scale. Where accuracy is required in a scale reading, however, recourse must be had to a vernier, or measuring microscope, where the recognition of coincidence or equality only is generally called for. The brain can recognize apparent coincidences of events in time, coincidences and contiguities of objects in space, and many equalities of sense impressions (such as brightness or loudness) to a greater or less degree of accuracy. Most instruments aim at reducing the human part of the measurement to those things of which the human subject is really capable.

For example, in measuring a refractive index by the spectrometer, the measurement is effected through the recognition of the coincidences of the cross wires with certain optical images, and of the coincidences of certain scale and vernier divisions.

A few physical measurements depend on the recognition of the " threshold " or beginning of a sensation when the stimulation increases from zero, but not many.

An instrument often provides a known or measurable amount of something to compare with an unknown amount of something similar. An unknown angle, such as that subtended between the sun and the horizon, may be compared with an angle measured on the divided circle of a sextant, the comparison being effected (through optical means) by the observation of several sets of coincidences. The photometer provides a measured amount of " brightness " to compare with an unknown brightness. In fact, the use of any instrument may thus be analysed, and the essential features of the experiment or measurement discovered.

In another method of action, typified by the ordinary balance, a displacement produced by an unknown force is annulled or reduced to zero by the action of another force which is known. This is a typical example of a *null method*.

Many instruments provide in some way (not necessarily by optical means) a magnification of a displacement or other phenomenon. This *may* increase the sensitiveness of the instrument at the expense of its accuracy, but on the other hand the magnification may be essential in order to produce a sufficiently great impression on the senses. Thus in a galvanometer the controlling force on the moving coil or needle may be so greatly reduced as to render the sensitiveness very high, while the indications of the instrument are almost valueless on account of the increased period of the vibrations. Again in a range-finder the optical magnification is absolutely necessary to render visible the small parallactic displacements of the images. Even in optical magnification, however, it is easily possible to go too far. Thus it is often convenient to employ a travelling microscope to measure the intervals between the lines in a photograph of some spectrum, but if the magnification is made too high the image of a line in the field of the microscope becomes broad and "fuzzy", with the result that the accuracy of setting is decreased.

Sensitiveness and Sensitivity.—While the term sensitiveness is used in a somewhat loose manner, the *sensitivity* is defined as the rate of change of the indication α of the instrument with respect to the change of the measured quantity F.

$$\text{Sensitivity} = \frac{d\alpha}{dF}.$$

Errors.—The use of optical and mechanical magnifying devices is very liable to introduce error because the magnified phenomenon is not the phenomenon itself. Thus it is frequently necessary to measure *images* of objects too small to be viewed directly, and in such a case the effects of optical aberrations (distortion and the like) have to be taken into consideration. Similar points arise when mechanical arrangements for magnification are considered. Taking, for example, the "screw", where small linear movements are connected with corresponding but large angular movements, it will be remembered that errors in the formation of the thread will cause variations in the proportionality of linear and angular displacements, thus introducing error into any measurements being made by this means. In the following chapters it will be necessary to consider instrumental errors and accuracy in some detail. We shall endeavour to show how far human errors and instrumental errors are involved in various measurements.

Mason * cites seven requirements regarding the accuracy of measuring instruments:

(*a*) An instrument must have the required accuracy.

(*b*) It must have a constant accuracy.

(*c*) Every important source of inaccuracy should be known.

(*d*) Any error should as far as possible be capable of elimination by adjustment.

(*e*) This adjustment should be possible without the addition of other special apparatus.

(*f*) When an error cannot be eliminated it should be made as small as possible.

(*g*) When an error cannot be eliminated it should be capable of measurement by the instrument itself without external apparatus, so that the results may be corrected.

Such requirements as the above will not be fulfilled in the design of instruments without great care and a thorough knowledge of the performance of optical and mechanical parts. Whipple recommends the attacking of a designing problem from the historical standpoint, so that previous efforts in the same direction may be studied and the reasons for their success or failure understood.

Optical Design.—The majority of the instruments to be considered in this book carry microscopes, telescopes, or other optical systems as fundamental features, and, as was previously stated, a part of the measurement is generally the observation of a coincidence in the field of view of the instrument. In some cases (for example, the microscope with eyepiece micrometer) the image itself has to be measured, but most often a definite axial image formed by the telescope or microscope objective is to coincide with " cross wires " in the focal plane. This coincidence is to be observed with a magnifier or eyepiece, which must produce a sharp retinal image of both cross wires and coincident images. The function of all the optical parts is to render aid to the eye, and the limitations and possibilities of that organ must receive primary consideration. It will be more convenient in this connection, however, to discuss each instrument separately, and limit ourselves here to a short description of the more usual aberrations of optical systems and their possible effects on images.

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* *Dictionary of Applied Physics,* Vol. III, p. 446.

Whatever be the nature of the optical system it should, to be perfect, bring all rays from points in the object space to corresponding points in the image space. This is suggested for the case of a telescope objective in fig. 1 (*a*). As it is impossible to produce lens systems which are perfect for all and any conditions, it is necessary to use special and differing arrangements to produce images of distant and very near objects respectively; for example, telescope and microscope objectives. In either case, however, the lens should give almost perfect ray congruence for point objects in positions suitable to the particular system.

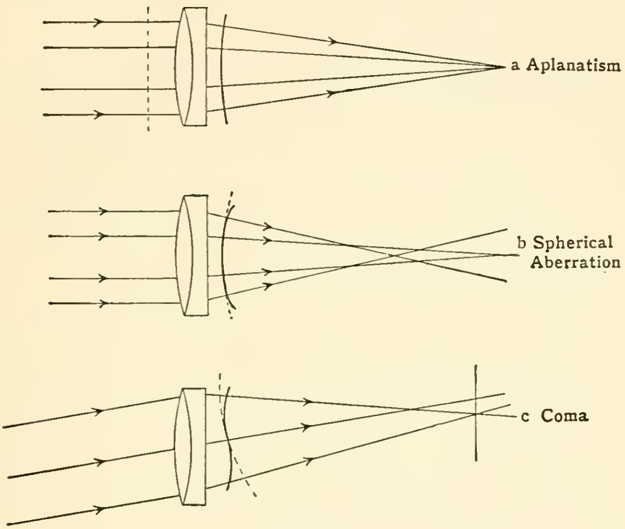

Fig. 1

For monochromatic light and for axial object points some modern lenses do, in fact, attain very nearly to the above condition of perfection, but the wave nature of light causes a spreading of the light into a small area surrounding the image point. Thus the image of a star formed by a telescope objective is really a diffraction pattern of the appearance suggested by fig. *a* (frontispiece), in which the circular central condensation of light is seen to be surrounded by rings of rapidly decreasing intensity. The radius of the first dark ring can be shown to be

$$r = 1{\cdot}22\frac{\lambda f}{a},$$

where λ is the wave-length of light, f the distance from the lens at which the image is formed, and a is the diameter of the lens aperture.

The formula also holds for images of extremely small bright objects formed by a microscope objective. A common aperture ratio (f/a) for telescope or microscope lenses is about 15/1, so that taking λ (for the mean wave-length of visible light) as 0·00005 cm. we find the radius of the first dark ring to be 0·0009 cm. This is very small, and, in fact, an eyepiece of considerable power is necessary to see the diffraction pattern at all.

Under the best conditions a fraction only (0·839) of the total light goes into the central patch, while the remainder is found in the outer parts of the ring system. The distribution of light was first computed by Airy, and the central condensation of light is often referred to as the Airy disc.

When the congruence is perfect it is possible to imagine the light waves derived from the object point as of truly spherical* form on leaving the lens, as suggested by the trace in fig. 1 (*a*).

Spherical Aberration.—Imperfect construction of the lenses, or their employment under wrong conditions, may, however, impair the ray congruence and the sphericity of the wave surfaces. In such a case, where the effect is symmetrical with regard to the optical axis, more or less general illumination or stray light is distributed round the central bright Airy disc, which is not, however, greatly increased in size. By altering the focus it is usually possible to find another more uniformly bright patch (the disc of least confusion), rather larger than the above disc, in which most of the light is found. Such effects indicate the presence of *spherical aberration*. The defects are shown, enormously exaggerated, in terms of ray paths in fig. 1 (*b*) and in the distribution of light in fig. *b* (frontispiece).

Keeping the discussion to the case of monochromatic light only, and assuming that the lenses are perfectly centred and spherical, it will be understood that the various forms of spherical aberration, in which the wave surface may be distorted (but still symmetrical round the axis), are the only defects which axial images can suffer. In the non-axial image points, however, other defects may be found.

Coma.—Fig. 1 (*c*) shows the type of ray distribution in the presence of the defect known as coma, which results when the lens system does not fulfil the optical " sine condition ".† In this case the beginnings of the defect are seen in the star disc by a weakening of the ring system to one side and its strengthening to the other side; at the same time the point of maximum intensity suffers a displacement (radial in the field) towards the side on which the rings are the stronger.

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* Saucer shaped, not complete spheres.

† E. T. Whittaker, *Theory of Optical Instruments*, p. 34.

Fig. *c* (frontispiece) shows a photograph of a star disc when coma is present to a slight extent. When heavier amounts are present the appearance is more like a comet with its tail on the side in which the " ring " appears in the figure, and thus more nearly resembles the appearance forecasted by geometrical theory. In the majority of telescope and microscope lenses every effort is made to eliminate coma; this is especially important in astrographic objectives, and, in fact, in any lenses with which it is desired to secure an image which remains sharp at a few degrees from the axis. If coma is present any defect in the centring of the lens will cause loss of definition in the centre of the field. It is, however, possible to secure other advantages by neglecting the fulfilment of the sine condition, so that in some types of telescope objective a certain amount of residual coma is tolerated. The advantages lie mainly in the greater ease of production and consequent cheapness. When coma is tolerated it is possible to use a double concave flint lens in the achromatic doublet. Concave surfaces are more easily made accurate than convex, more especially in the larger lenses.

In order to illustrate the effect which this aberration produces upon an image, the following more definite particulars may be of interest. In the presence of coma, the displacement of the point of maximum intensity from the intersection of the central ray in the image plane is roughly equal to the radius of the Airy disc when the maximum optical path difference between disturbances arriving from the wave surface is equal to two wave-lengths. An old-fashioned type of doublet astronomical object glass, with an approximately equi-convex crown lens, is likely to possess an offence against the optical sine condition of one part in a thousand with an aperture ratio of 15/1. The path difference in the image plane then rises to two wave-lengths at 1·77 cm. from the axis. With a 12-in. objective of such a type, having a focal length of 457 cm., this corresponds to an angle with the axis of approximately 13′. The displacement of the point of maximum intensity from its proper position may only be roughly proportional to the distance of the image from the axis. For a more complete discussion of the physical effects of spherical aberration and coma, reference may be made to papers by Professor

Conrady * and the present writer.† Coma is especially avoided in the construction of astrographic objectives because it produces a displacement of the image unsymmetrical with regard to the central ray. A good number of hitherto unexplained discrepancies in the indications of the positions of stars by photographs made with certain instruments may be laid at the door of coma.

Should the lenses in a compound system be relatively badly centred, coma will appear on the axis.

Astigmatism.—If coma is successfully eliminated there still remains the defect of *astigmatism*, which afflicts the majority of telescope and microscope objectives in images away from the axis. It is generally manifest near the edge of the field of view in telescopes and microscopes, when observing " star disc " images, as an elongation of the circular Airy disc into a short line which is radial or tangential (with reference to a circle with its centre on the optical axis) according to the focus. Between the two positions a more or less circular patch of light is found which may be somewhat more diffuse than the Airy disc. Fig. *d* (frontispiece) suggests the appearance of the Airy disc when modified by a small amount of astigmatism. The effect is due to variations, along different meridians, of the curvature of the wave surface leaving the lens. The defect often cannot be eliminated simultaneously with other aberrations, except by the use of suitably placed diaphragms,‡ as is done in photographic lenses, and therefore it has to be tolerated in the majority of telescope and microscope objectives. It does not, however, produce a displacement of the image which is unsymmetrical with regard to the central ray.

Curvature of Field.—Should we examine a telescope objective showing astigmatism, and investigate the location of the star images up to, say, 10° from the axis, it would be found that the radial images lie roughly on a portion of a curved surface, and the tangential images on another curved surface touching the first at the axis, but of different curvature. At large angles with the axis other aberrations begin to be troublesome, and it is no longer possible to analyze the effects as simply as we have done up to the present.

The images will therefore not be all in focus on a flat plate, although by choosing a suitable position for the plate we may obtain a fair " field " in which the focus is sufficiently good, since all lenses have some " depth of focus ".

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* *Monthly Notices, R.A.S.*, June, 1919; Jan., 1920.

† *Ibid.*, March, 1922.

‡ Whittaker, *Theory of Optical Instruments* (Cambridge Tracts), p. 39.

When astigmatism is eliminated the two focal surfaces will coincide, but the field may not yet be flat. Something can be done, however, to flatten the field of very complex systems by choosing suitable glasses for constructing the lenses and thus fulfilling the so-called Petzval * condition.

In optical measuring instruments the chief trouble of extra-axial aberration lies in the corresponding defects of the eyepiece, which has generally a curvature of field in the sense that a flat object viewed by means of it is not all in perfect focus. In the majority of cases the field curvatures for objective and eyepiece are in opposing senses, so that the actual portion of the image in satisfactorily sharp focus in a telescope or microscope is very small. Some latitude is possible, however, through the power of accommodation of the eye.

Fig. 2 illustrates the case of the curvature of field of a telescope object glass lens. The traces *t* and *s* represent the surfaces containing the tangential and sagittal images respectively, while *p* represents the so-called Petzval surface. Owing to this trouble it is necessary in some telescopic instruments, where a large field is to be examined, to provide means for traversing the eyepiece at right angles to the axis of the instrument, so that the central part of the eyepiece field may be always employed.

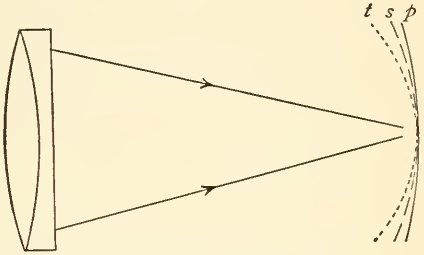

Fig. 2.—Curvature of Field (size of lens exaggerated)

Distortion.—The effects of this aberration are familiar to users of cameras with simple landscape lenses in which a stop is used in front of a concavo-convex meniscus lens. A few simple experiments with a camera and a lens showing distortion will soon elucidate the nature of the defect. Let a long scale be photographed so that its image completely stretches over a diameter of the plate. It will then be found that if the *images* of two intervals are measured (one short and one longer interval, each having its centre at the centre of the plate), an interval which on the actual scale is, say, four times the length of the shorter one will not bear this ratio in the image. It may be less than four times (" barrel distortion ") or greater than this (" pincushion distortion "). The defect arises from inconstant magnification for varying angular size of field, and the defect is roughly proportional to the cube of the radius of the field (contrast this with coma, in which the magnification varies directly with the radius of the field).

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* Whittaker, *Theory of Optical Instruments* (Cambridge Tracts), p. 39.

Distortion does not lead to defective "definition" of the image in the absence of other aberrations, but we need to be assured in measuring photographs that distortion is negligible before assuming that the dimensions of the object are accurately proportional to those in the image. Mere inspection of images is entirely inadequate to reveal the defect, as the eye is quite insensitive to distortion, even though it may be present in the images given by telescopes and microscopes *as seen through the eyepieces* to a very measurable extent. Fig. 3 illustrates distortion and how it arises from inequality of magnification for different angles of the field of view.

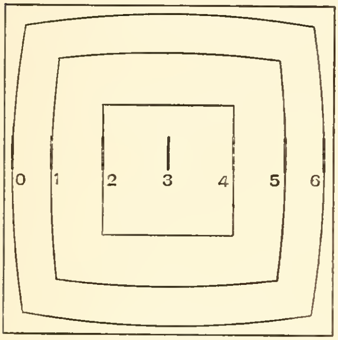

Fig. 3.—Distortion

The "first images" formed by telescope objectives are for all practical purposes quite free from distortion, and the same remark applies to the images formed by the simpler "doublet" microscope objectives. With more complex objectives it would be as well to search carefully for evidences of its presence if measurements of the image are to be made for very special work. The effects are, however, bound to be extremely small. This does *not* apply to the virtual image as viewed through the eyepiece, for distortion of considerable amount is introduced by the majority of oculars. Thus a scale in the focal plane of a Huygenian eyepiece (between the lenses) would not give satisfactory indications of the relative proportions of the parts of the object, the images of which are being viewed apparently in coincidence with the divisions of the eyepiece scale. For this reason Ramsden eyepieces are invariably employed for micrometer eyepieces. The measurements are then made directly on the first image formed by the objective, whether in telescope or microscope.

Chromatic Aberration.—A simple double convex lens is found to bring the shorter wave-lengths of light to a focus nearer the lens than the focus of the longer wave-lengths.

If a star image formed by such a lens (with white light) is viewed by a magnifier, we may find at one extreme a violet and blue focus with surrounding rings of green and red (this at the focus nearest the lens) and at the other extreme a red focus with surrounding rings of green and blue. With an *achromatic* lens the appearances are quite different, the colour effects at the star focus being much less marked. In the first case we had something resembling an axial spectrum formed by the variously coloured rays focusing at distances from the lens increasing with wave-length; but in the second case (the achromatic lens) this spectrum is shorter and *folded*, so that when the correction is best, the yellow-green rays come to a focus at the point nearest to the lens, while the red and blue-violet are focused at a greater distance. Fig. 4 illustrates diagrammatically the variation of the focusing point with wave-length for an achromatic lens. On looking at the star with a magnifier or eyepiece as before, a comparatively white focus is seen, but inside the focus an expanded disc with a faint red outer ring is generally formed, while outside the focus there will be a disc with red-violet centre and a greenish outer portion.

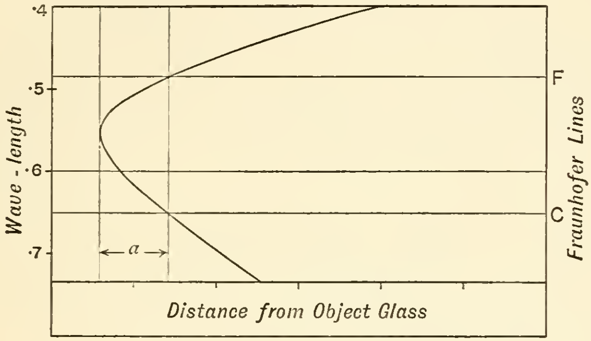

Fig. 4.—Chromatic Aberration

These effects are of the type seen when using a " doublet " achromatic lens, which is by far the most usual type in optical measuring instruments. In some telescope and microscope objectives, however, " triplet " lenses may be used still further to improve the chromatic corrections.

Referring back to the discussion of the Airy disc, it may be quickly found from experiment that the concentration of the light in the disc remains well marked for a definite interval on each side of the focus, in which interval the disc does not increase in size or lose much light. The condition for successful achromatic correction is that the spreading of the various foci for different wave-lengths shall not be so great as to prevent a satisfactory union (possibly through the range, depth, or toleration of focus) of the Airy discs corresponding to the range of visible wave-lengths of light.

General Inferences.—The foregoing remarks should make it clear that optical images must be used with caution and that, especially when the utmost refinements in measurement are aimed at, the possible effect of the aberrations which may be present should be known as accurately as possible. In astronomical measurements the methods adopted are often specially chosen to eliminate the effects of errors of this class. Thus, for example, in investigating the apparent angular shift of stars the light of which passes near the sun (the Einstein effect), it is of the highest importance that the same field of stars shall be photographed at about the same altitude with the same instrument when the sun is not present at another time of the year. It would be hopeless to try to compare the indications of one photograph with measurements made by other means.

Mechanical Design of Instruments.—The first writer to apply the theoretical considerations of kinematics especially to the case of instruments was Clerk Maxwell,* and we may follow his line of thought with advantage.

Every unconstrained particle of matter in three-dimensional space has three degrees of freedom, or three mutually perpendicular directions of motion. Every rigid body (or grouping of particles in a non-variable shape) has also three similar translatory degrees of freedom when unconstrained, and, in addition, three independent possible rotations about three mutually perpendicular axes.

If a rigid body is held or constrained, one or more degrees of freedom may be destroyed, and Clerk Maxwell states that if a solid piece is constrained in *more* than six ways it will become subject to internal stress.

These ideas become greatly clarified when simple concrete examples are taken. Consider the case of a three-legged stool standing on a flat but, say, frictionless floor. It has lost one " translation " and two rotations. (We assume that gravity is sufficient to prevent any of the legs from leaving the floor.) Clearly, however, it may still be moved in two mutually perpendicular horizontal directions, and it can rotate about a vertical axis.

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* *Handbook of the Special Loan Collection of Scientific Apparatus in 1876.*

If one leg, or rather a point in one leg, is then fixed, the two remaining translations are destroyed, and the body possesses only one degree of freedom, namely, the rotation about the perpendicular axis through the above leg. When two legs are fixed all degrees of freedom are lost.

It may be noted that a fourth leg for support is redundant to a really rigid body, but in the case of such articles as tables there is usually sufficient elasticity in the material to produce, under the force of gravity, deformation sufficient to bring four legs into contact with the floor. Four legs of course are preferable for such a rectangular article, as the possibility of overbalancing is minimized. In some stands an adjustable screw is provided, which affords a means of altering the length of the fourth leg.

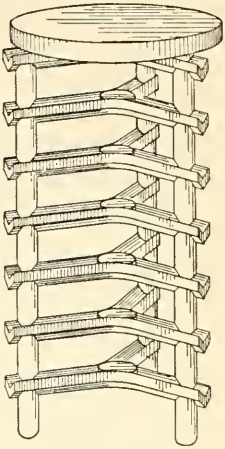


Fig. 5.—Geometric Stand (C. V. Boys)

Three-legged supports are usually employed in instruments (except in certain forms of the theodolite), and this suffices to produce a non-shaky mounting. The geometric apparatus stands of C. V. Boys are well known and their principle will be obvious from the inspection of fig. 5. The stands are built up from a number of units (seven are shown in this figure) each having three hemispherical feet corresponding with V grooves in the upper surface, each foot having two point contacts in each V, so giving the necessary and sufficient six points of constraint. The levelling screws of any instrument may be placed in the grooves or on a table at the top.

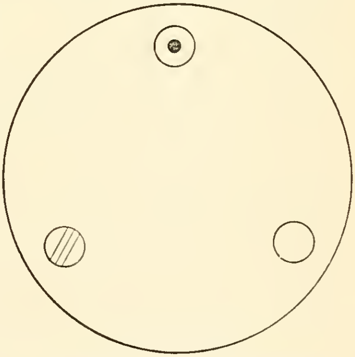


Fig. 6.—Hole, Slot, and Plane Mounting

Given an instrument with three hemispherically ended feet, a convenient method of mounting is to employ the "hole, slot, and plane" device suggested in fig. 6.

One leg rests in a conical hole made in a flat surface, while another rests in a V slot of suitable width pointing preferably towards the first hole. The third leg rests simply on the flat surface. Under these circumstances it will be seen that no lateral strain on the legs is caused by the mounting, such as would be the case if the three legs were caused to rest in three holes the relative distances of which were not absolutely identical with those of the legs when unstrained. This method of mounting is of frequent use when it is desired to be able to replace an instrument frequently in a constant position with regard to some other apparatus. The mounting will not, however, be quite truly geometric unless it is arranged that contact between leg and " hole " shall occur only at three points, as, for example, by shaping the hole as a triangular pyramid. The points of constraint are now six in number. There are three point contacts of the hemispherical foot in the triangular hole, two point contacts between the second foot and the sides of the V groove, and one point contact between the third foot and the plane. The resultant of the upward forces acting on any leg is vertical (being just sufficient to oppose the downward gravitational force), and there can evidently be no resultant lateral force on any of the legs. It is sufficient to consider the direction of the force through a point contact as being normal to the surfaces at the point.

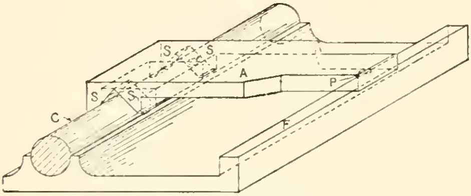


Fig. 7.—Geometric Slide

Owing to the physical properties of materials the ideal conceptions of point constraints (as consideration of the above examples will show) are never realized in practice, but, in general, constraints are exercised over areas of varying sizes and shapes. Where large forces are met with large areas must be used. Nevertheless the " geometric " ideas seldom fail to point in the right direction, and help greatly in the production of designs which avoid internal strains in instrumental parts.

One example of geometric design is exemplified in the slide shown in fig. 7, in which an accurately formed cylinder plays an important part. The sliding portion A has two V's on its under side, and studs (marked S in the diagram) are mounted in them. The four studs S are in contact with the cylinder, while another stud P rests on the plane F, which is parallel to the axis of the cylinder. The only motion possible to A is therefore one of translation parallel to the axis of the cylinder.

In an alternative design the part A can be rigidly fixed to the rod C, which latter rests in two V's with studs on them. This is essentially the method employed in the Cambridge measuring microscope to be described in the next chapter. If the studs are supposed to have slightly convex surfaces the constraints on the moving part are now seen to be applied at five points, and only one possibility of motion is left. If the stud P were hemispherical and rested in a groove perpendicular to the axis of the cylinder, then there would be six points of constraint and all degrees of freedom would be destroyed. If a force were then applied to the "moving part" at any other point, internal stress would be set up.

It is necessary carefully to guard against misconception of the purely geometrical ideas of Maxwell. *The idea of constraint between a stud and a cylinder involves nothing more than the fulfilment of the condition of contact between these parts*, and the whole of the ideas are essentially independent of any ideas regarding the weight of any mechanical parts, or the action of gravity. It will be instructive to consider the foregoing examples with these last words in mind. Especially, too, one must guard against the tendency to assume that unless an object is held in more than six places it cannot be subjected to internal stress. In point of fact when actual instruments are manufactured it is often impossible to avoid more or less internal stress through the weight and other causes.

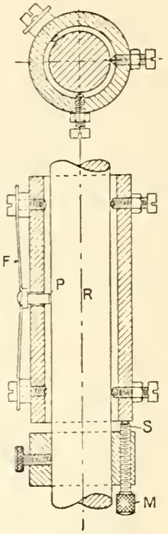


Fig. 8.—Cathetometer Mounting

It is naturally not always possible to rely entirely on gravity for the constraint of parts intended for use in practical instruments, as by far the majority of instruments have to be used in varying positions and have to be able to be carried without falling to pieces. In such cases springs are finding an increasing usefulness, and an example is given by a recently constructed measuring microscope, of which the design embodies something of the principle of the slide in fig. 7. In this case a long slot is cut in the base, through which passes a rod rigidly fixed to the member A. By means of a sliding spring a constant downwards tension is exerted through the rod on A. Fig. 8, again, exemplifies a mounting for a cathetometer telescope in which the spring F keeps four screws (projecting into the interior of the tubular telescope carriage) in constant contact with the surface of the vertical pillar R. The carriage is thus able to take up its position without strain, and there will never be sticking between the rod and carriage, such as often occurs between a rod and a too closely fitting sleeve.

While the geometrical principles of design are very useful in indicating the method of approach to various problems, a limit is set to their utility by the properties of materials. In the next and subsequent chapters the various modifications of the geometric constructions made necessary in practice will be illustrated. Point and line constraints have to give place to surface constraints, and the heavier and more solid moving parts of instruments call for the greatest departures from the theoretical ideal. Permanence of adjustment is very often of great importance, and with small bearing surfaces the amount of wear is often very considerable. On the other hand it is often possible to ensure by geometric design that the wear which does take place does not lead to loss of adjustment. Wear can be controlled by increasing the area of the bearing surfaces and thus decreasing the intensity of the pressure between them.

CHAPTER II

Optical Pointers and Measuring Microscopes

Optical Sights and Pointers.—Sights and pointers have been used since very early times for shooting and for astronomical purposes. In certain ancient types of altazimuth instrument (used up to about A.D. 1650), the pointer consists of a bar carrying at one end a vane with a very small aperture and at the other end a small metal bead or point. The eye looks through the hole and the point is brought into apparent contact with the object as thus seen. Another simple example is the "aperture and line" sight, such as is employed in prismatic compasses (fig. 9), and in cases where azimuth angles only are required.

The modern device for accurate "sighting" purposes is a lens system carrying in its optical axis a cross wire or other pointer, and this is made to coincide with the image of the object (fig. 10). The coincidence is viewed with a magnifier forming the eyepiece of the resulting telescope or microscope. Accurate focusing is necessary to prevent possible parallax (and consequent inaccuracy of setting) between cross wire and image. A little consideration will show that there is only one direction of the optical pointer for which the

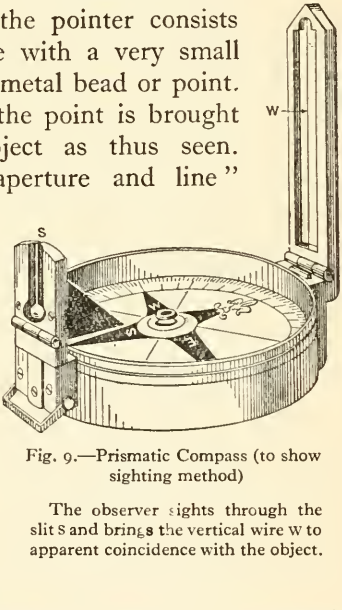


Fig. 9.—Prismatic Compass (to show sighting method)

The observer sights through the slit S and brings the vertical wire W to apparent coincidence with the object.

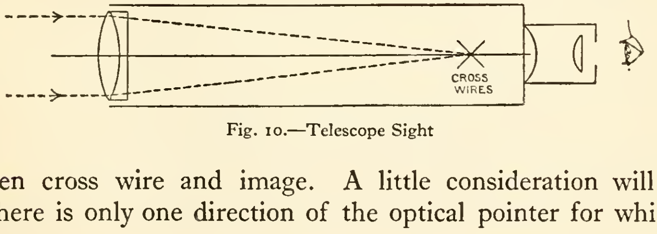


Fig. 10.—Telescope Sight

image of the object "sighted" will be located on the intersection of the cross wires. Consider a ray starting from this intersection and travelling to the second nodal point of the lens. It will continue its path beyond the lens as if it came from the first nodal point, and it will make an equal angle with the axis. Should the optical pointer be turned, this ray path will no longer intersect the object, and consequently the image cannot be formed in its original position. The line joining the cross-wire intersection to the second nodal point of the lens system is often called the *line of collimation* of the optical sight.

Optical sights for astronomical observation were introduced by Gascoigne, and the new method was strongly supported by Hooke; this was the occasion of a great controversy with Helvelius of Danzig (1611–87), who maintained the equal efficacy and accuracy of the hole-and-pointer sight, which could be made to give results of higher precision by using the four sides of a square disc as the foresight, and four corresponding slits at the sighting point. When "set" on a star the star had to appear exactly at a side of the disc as seen through any one of the slits. The English astronomer Halley paid a special visit to the observatory of Helvelius to test the relative accuracy of the methods of observation. The controversy may appear surprising until the optical defects and imperfections of the early telescope are remembered. The high attainments of modern instruments have, however, placed the superiority of the optical pointer beyond all question.

In the designing of the lenses for an optical sight it is desirable, as far as possible, to correct both objective and eyepiece separately for spherical and chromatic aberration instead of balancing the defects of the one by the construction of the other, as is often the case in ordinary instruments. Instruments confining their use to settings in the centre of the field may, by departure from the usual constructional types, have the Gauss condition fulfilled, and the spherical aberration of the objective may thus be corrected for two wave-lengths of light. It is doubtful, however, whether this is a sufficiently great advantage to compensate for the (then unavoidable) presence of coma. Coma must be eliminated as carefully as possible in objectives intended for use with micrometer eyepieces. Absolute correction of the objective tends to minimize probable parallax between image and cross wire, and any residual aberration in the eyepiece affects the image and pointer in the same way.

Optical sights and pointers form the essentially optical part in a very wide range of instruments: measuring microscopes, theodolites, transit and other astronomical instruments, spectrometers, and the like. Directional telescopes are of all sizes and magnifications, but measuring microscopes have usually magnifications between 20 and 50 diameters and 1-in. to 4-in. objectives, although higher powers can, of course, be employed for special purposes. There are other instruments such as cathetometers in which the construction of the optical parts may be intermediate between a telescope and microscope.

Measuring Microscopes.—We shall first of all deal with the simple laboratory or workshop instrument intended for length measurements with an accuracy from 0·01 to 0·001 mm. For simple length measurements the optical pointer is usually made to move at right angles to, or along, its axis. It moves on slides, and the motion is registered either by a scale and vernier or by a micrometer screw. In all such instruments the support must be *broad* and strong to render the direction of the pointer more independent of small local irregularities in the slides, and to prevent shakiness. Some suitable supports were discussed in dealing with geometric designs in Chapter I. Where a cylinder is not employed a V-shaped slide is often made in the casting for the mount. Two V projections fitting the V slide are then formed on the carriage, which is further supported by a stud sliding on a planed surface at the back of the mount. As the arrangement is still otherwise similar to that suggested in fig. 11, the mount is still semi-geometric, while greater stability and strength are secured. This type is illustrated in fig. 11.

An example of the employment of a cylinder resting in V's is shown in fig. 17, which illustrates the Cambridge measuring microscope. The stop which prevents the cylinder from rotating will be recognized. It is kept in contact with the straight piece by means of a cantilever spring also attached to the end of the cylinder. More particular details of these instruments are given below.

Sources of Instrumental Error.—In such simple types of instruments the possible sources of instrumental error may be summarized under five heads.

1. Bending or warping of the slide or the mounting of the pointer.

2. Errors in the scale or vernier (vernier instruments).

3. Errors or irregularity in the pitch of the screw (micrometer screw instruments).

4. Periodic errors due to bad shaping of the shoulder or bear-

ing of the screw, or from failure to have the thrust bearings perpendicular to the axis of rotation, or to dirt in the bearings, or to inaccurate division of the circular head of the instrument (micrometer screw instruments). Periodic error may also arise through eccentricity of the screw axis with respect to the axis of rotation, and it is possible that eccentricity of the drum head might give a similar error.

5. Imperfect focusing or unsuitability of optical parts.

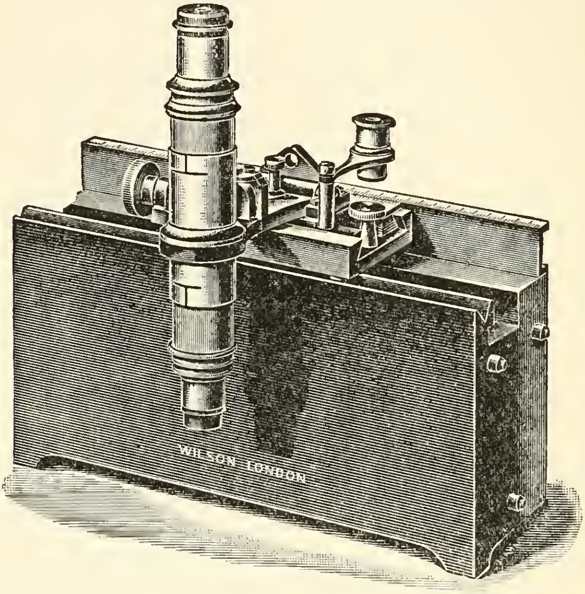


Fig. 11.—Laboratory Vernier Microscope by Wilson

These are, of course, in addition to the personal errors arising from inaccurate setting.

It may be stated at once that with reasonable care the instrumental errors are generally negligible with *well-made* instruments in cases where an accuracy of 0·01 mm. is sufficient; that is, of course, unless the instrument has been badly worn or damaged.

For many purposes, however, "absolute" results are required to an accuracy of 0·001 mm. or 1 μ, and, in order to attain this end, the most careful consideration of the instrument and method of procedure is necessary. We may proceed to consider separately the items under the above heads, and afterwards to review means by which the errors may be ascertained and allowed for.

Form of Slide.—In fig. 12 (*a*) P_1 and P_2 represent two positions of an optical pointer fixed on a mount M which moves on a warped slide S. In the two positions shown the pointers are directed to, or " set on ", the marks A and B below, but, owing to the non-parallelism of the two directions of the pointers, the displacement of the mount is not equal to the distance between the marks.

Supposing that the distance between the bearings of the mount on the slide were roughly equal to the perpendicular distance from mount to object, it will be seen that the error introduced would be comparable with the depth or height of any irregularity on the slide. Now the irregularities on optically polished (so-called flat) surfaces may be of the order of say 0·0001 mm., but on the machined slides of the instrument much larger errors may be encountered, and certainly 0·001 mm. would not be unexpected.

It will be seen that the trouble might be largely eliminated by arranging that the object shall lie in the plane of the slide and providing the necessary higher support for the microscope tube.

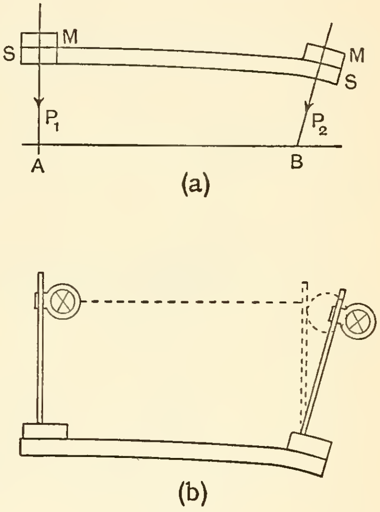


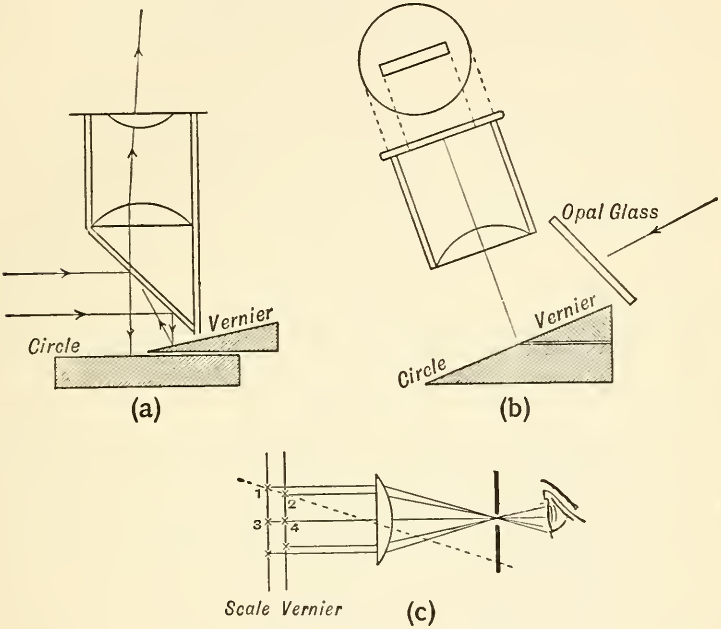
Fig. 12

This arrangement, however, does not make for compactness and has not been adopted in many instruments, although it would seem worth while because any wearing of the slide would then cause the minimum effect. That the possibility of error from this source has not been recognized in one case was made sufficiently clear by the provision of a tall pillar on the carriage or mount, the telescope being fastened perpendicular to the pillar at some elevation in order to obviate the necessity of raising the heavy base of the instrument when the object to be measured is at some height above the bench. Such an arrangement is highly unsatisfactory except for very rough work. Fig. 12 (*b*) will make the point clear. The displacement of the axis of the optical pointer is evidently much greater than that of the mount on the slide.

Scale and Vernier.—For the elementary theory of the vernier reference may be made to Watson's *Practical Physics*, p. 43.

It will be remembered that errors in the length of the vernier should be looked for, and to this end it is a help to have a few graduations divided beyond the zero and end marks of the vernier. The average length of the whole vernier with reference to the scale divisions can then easily be checked by taking observations at several points on the scale, and any necessary correction may then be found. The correction to a particular vernier reading is proportional to the corresponding distance from the zero to the coincidence point. As "scales" are to be considered more particularly in a later section, this subject hardly calls for further comment as far as measurements of length by the vernier are concerned. It is very seldom that an accuracy of a higher order than 0·005 mm. is aimed at by this means as the micrometer screw is more convenient and, above all, much easier to read. Something may be said with advantage, however, about the practical construction and illumination of scale and vernier.

In many types of instrument the vernier is engraved on a wedge of small angle sliding over the scale, and some indifferent magnifier or Ramsden eyepiece (with a reflecting device for illumination) is provided for reading (fig. 13 (*a*)). The result is generally that the graduations of scale and vernier are seen under vastly different conditions of lighting. Each division is a sleek, mark, or smooth scratch of some form, generally in a silver or white-metal surface, and sometimes the edges of scale and vernier divisions may reflect light in a most unequal way. With arrangements such as the above the present writer has been able to produce apparent variations of three divisions in the coincidence point merely by altering the direction of the light. A far better arrangement is shown in fig. 13 (*b*). Vernier and scale are in the same plane. The lighting is from a diffusing surface of milk glass and the reading is taken by means of a lens with a "telecentric" stop or slit. This device, which is finding increasing use in optical instruments, consists of an aperture placed at the principal focus of a lens, through which the eye is to observe the image of the object to be magnified. The axis of the lens must in the above case be *held* perpendicular to the plane of scale and vernier. Referring to fig. 13 (*c*), it will be seen that, under these conditions, only rays originally proceeding from the scale or vernier very nearly parallel to the axis of the lens can reach the eye. The line through points 1 and 2 in fig. 13 (*c*) (denoting divisions on the slightly displaced scale and vernier respectively) is not parallel to the lens axis; therefore a ray passing in this direction cannot reach the eye, as it might if the telecentric magnifier were not employed and if the eye were moved downwards a little. Parallax is thus eliminated, and only points such as 3 and 4, both on a line parallel to the axis of the lens, can appear to coincide. The length of the stop may clearly be enlarged in a direction parallel to the scale divisions in order to obtain more light.



(a) (b) (c)

Fig. 13

The Micrometer Screw.—The production of accurate screws is a subject which can hardly be treated at length in such a book as the present one. Those employed in accurate measuring instruments are cut on special lathes, but they can be further improved by a special process of interaction between a long nut, which is capable of being tightened, and the screw itself.

A fine grade of abrasive, usually rottenstone, is used between the threads, and the nut is worked up and down and gradually tightened. Irregularities are smoothed out under influences similar to those operating in the production of regular spherical surfaces by grinding between more or less irregular surfaces. While it is possible in this way to smooth out slight irregularities in the threads, it is not possible to alter, in a determined way, the absolute value of the pitch. Care is necessary during the operation, as the cut must not be forced lest the surface of the metal becomes swaged and an inconstant pitch results. The *form* of the threads is also of great importance in securing the best results, and standard specifications* must be adhered to. The greatest care is necessary in freeing the metallic surface from the abrasive after the grinding process.

Apart from deficiencies in the form of the thread, the pitch may exhibit a progressive or periodic change, the latter being known as " drunken thread ". The " feel " of the thread running through the nut from end to end conveys much information to a practised hand. While a progressive change of pitch would tend to give increasing error in different parts of the thread, drunken thread might cause comparatively large error in measuring displacements which are only fractions of a millimetre, that is with a screw of 1 mm. pitch.

Second only in importance to the thread of the screw in a measuring microscope is the bearing. The arrangement varies in different instruments. In some the bearing is a spherical ball rotating in a corresponding spherical cavity. This is generally the case when the nut is the moving part and the screw merely rotates. In other instruments the nut may be stationary while the screw progresses, as in the *Cambridge* measuring microscope previously mentioned, where a small cylindrical steel rod with hemispherical ends bears in two corresponding cavities in the screw and the end of the cylinder. This is shown in fig. 18. The bearing, whatever its nature, must be kept scrupulously clean, and, this being the case, the error found with well-ground and accurate surfaces should be very small. The foregoing remarks apply to all cases where a screw is required to secure a definite linear displacement by a definite angular rotation, and thus they apply equally well to the action of the leading screw in a lathe, which may produce incorrect screws owing to its own deficiencies and to those of the bearings and gear wheel.

Griffiths cites the following causes as the most frequent in causing errors in the screws produced in the lathes:

1. Intrinsic error of the leading screw.
2. Incorrect centring of the gear wheel on the lead screw.
3. Lead screw not revolving truly about its axis.
4. Imperfect adjustment of the bearing which takes the end thrust.

* *Report of Committee for Standardization of Elements of Optical Instruments*, Board of Industrial Research, 1920.

The screw in a measuring microscope is almost always of steel, while the nut is of brass or even some softer alloy. Wear of the nut is of very little moment, as there is generally a certain amount of backlash which has to be taken up before the nut moves unless, as in the Cambridge microscope, a nearly constant pressure is maintained between the nut and threads by the action of a spring.

Inaccurate division of the micrometer head is seldom a matter for trouble, but its possibility should be borne in mind when investigating any mysterious discrepancies which seem unattributable to any other cause.

Hysteresis.—Effects of lag or " hysteresis " are very commonly encountered in the study of the properties of materials subjected to stress. The numerical amount of the response to a given change of stress may vary with the sign of the change. In the case of magnetism, where saturation is experienced, the characteristic hysteresis curve (a closed loop) is well known.

Effects varying with a direction of displacement are encountered in the study of certain instruments. If a micrometer screw be tested against an accurate standard scale in both backward and forward directions, the resulting closed curve obtained by plotting indications of the screw against scale readings will clearly exhibit the backlash as well as the other progressive and periodic errors of the screw and nut for both directions. Such a curve would be said to exhibit the hysteresis of the instrument. It should be noted that the periodic errors of a screw (owing to the change of the bearing surfaces) will seldom be the same in both directions of motion unless the backlash is taken up by a spring. Even then the pressure between bearing surfaces tends to differ between the two conditions of progress, consequently a residual hysteresis error may be detected even under these conditions.

Optical Arrangements and Adjustments.—When the image is focused accurately in the same plane as the cross wires no effects of parallax can cause apparent coincidence, and coincidence is therefore not seen unless the setting is correctly made. In the case of inaccurate focusing, however, the setting may easily be misjudged through parallax, as will be appreciated by reference to fig. 14.

The liability to error in this respect can clearly be diminished by using an objective of narrow aperture. The limits to the useful decrease of aperture are set, of course, by the decrease in the illumination of the image and by any loss of definition owing to the increased size of the elementary diffraction discs composing the image. The latter will seldom be a valid consideration until the aperture is reduced to, say, a diameter of 3 or 4 mm., and, granting the possibility of obtaining sufficient light, there appears no reason to use objectives of such large aperture as are commonly employed for purely observational purposes.

Sometimes it is difficult to arrange a long object to be measured so that its whole length is in perfect focus, or it may be desired to compare objects in slightly different planes. In such a case the aperture of the objective may often be reduced with advantage, and the best method is to place a telecentric stop or slit in the back focal plane. The diagram (fig. 15) will make this clear. Just as in the previous case of the simple magnifier, no oblique ray, such as might touch the two ends of the objects A and B and cause an apparent coincidence of images, can reach the image plane. The slit is always parallel to the graduations. It has the advantage that a line on the object perpendicular to the graduations may be focused with the full aperture of the lens, and the whole image is correspondingly brighter than when a circular stop is employed, while the correct focus is easier to find.

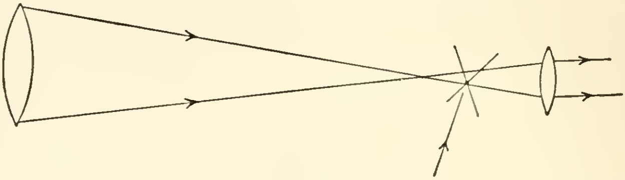


Fig. 14

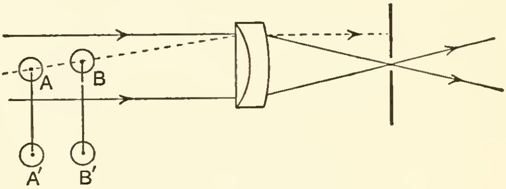


Fig. 15.—Telecentric Stop

Cross Wires for Measuring Microscopes.—The cross wire of a measuring microscope may take many forms, but there are certain types possessing special advantages in particular cases. Fig. 16 shows several cases. In Nos. 1, 2, and 3 the case of a somewhat diffuse line, such as is often found when measuring a photograph of spectral lines, is shown. No. 1 shows a most undesirable type of setting, one cross line being practically hidden. No. 2 is better, but not good in this case. No. 3 is best of all, but the separation of the parallel lines needs to be adjusted to suit the particular case. The slightest movement to one side or the other should show light between the dark image and crossline. When a very fine and sharp line (on a scale, for example) is to be observed, the type of setting in No. 4 is perhaps the best. Most of these settings are really photometric, that is they depend on the capacity of the eye to detect small amounts of light. Even the sense of symmetry probably depends on this in the last resort. A paper on "Recognition of Detail"* by S. D. Chalmers will throw light on this topic. The resolving power of the eye is limited to the resolution of objects separated by one or two minutes of arc in the field of view, but experiment has shown that where symmetrical settings are concerned, as in placing the two halves of a line in true alignment (No. 7, fig. 16), the probable error of setting involves an angular separation of only 10 to 20 sec. of arc for ordinary observers, and much less after training.

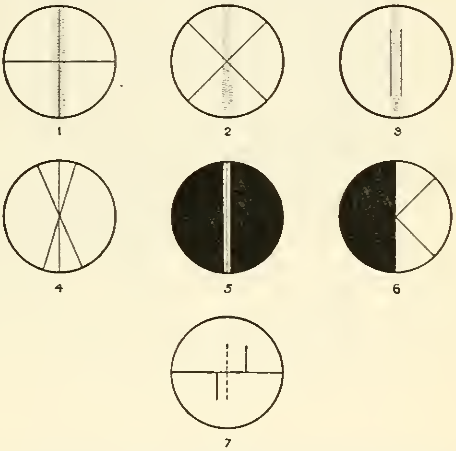


Fig. 16.—Types of Setting in Optical Instruments

The use of lines engraved on a glass (a *graticule*) is finding an increasing favour; spider lines are hardly sufficiently robust, at any rate for workshop purposes, and they are more difficult to renew. The primary conditions for accuracy of setting are a suitable brightness of field, a symmetrical result, and the absence as far as possible of disturbing appearances which mar the whole effect of perfect symmetry. In some cases an unsymmetrical setting is unavoidable, as in the Pulfrich refractometer. The case is suggested diagrammatically in fig. 16, No. 6. No. 5 in fig. 16 shows the use of a fine occulting wire for settings on a bright line such as is found in the use of the spectroscope.

* "Recognition of Detail", S. D. Chalmers, *Trans. Opt. Soc.*, Vol. XX, No. 9.

Reading Microscope (Precision Tool Company).—In this instrument a massive rectangular casting forms the base, which may be placed horizontally or up-ended. The form of the sliding parts has been discussed (p. 19). The microscope itself is mounted on an auxiliary slide provided with a fine adjustment screw. The vernier is fixed directly to the microscope mount, and both scale and vernier are in the same plane. A swinging magnifier is provided. Four positions are available for the microscope, in two of which its axis is parallel to the motion; in one position objects may be observed on an adjustable stage clamped to the front of the base casting.

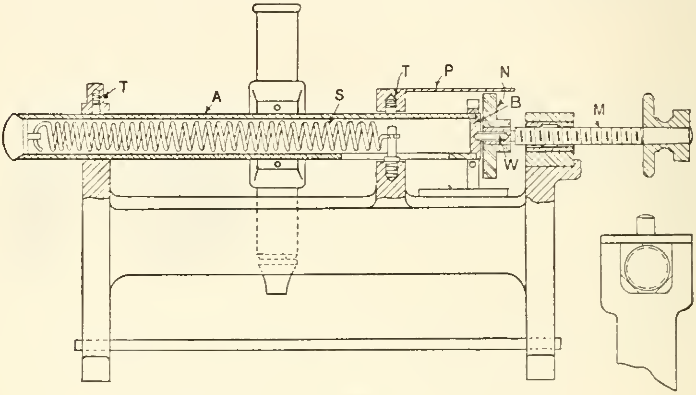


Fig. 18.—The Cambridge Microscope (diagrammatic)

A 2-in. objective is used with the microscope and the focusing is by hand. The instrument by Wilson illustrated in fig. 11 is similar and excellent in many respects.

Cambridge Measuring Microscope.—This instrument is shown in fig. 17 and diagrammatically in fig. 18; points relative to the general design of the instrument have already been discussed, but the strong spring in the interior of the cylinder should be noted, also the springs T (fig. 18) which hold the cylinder in contact with the V supports. In a recent form of the instrument the stationary nut is in two sections and the tightness with which these are held together can be adjusted. The bearing surfaces of the nut are of a comparatively soft white metal, and the two parts are held together by screws which are opposed by short spiral springs under compression. In order to understand the geometric nut, imagine that a cylindrical rod rests in two close V supports carrying studs, the contact being maintained by a spring stud pressing down on the top

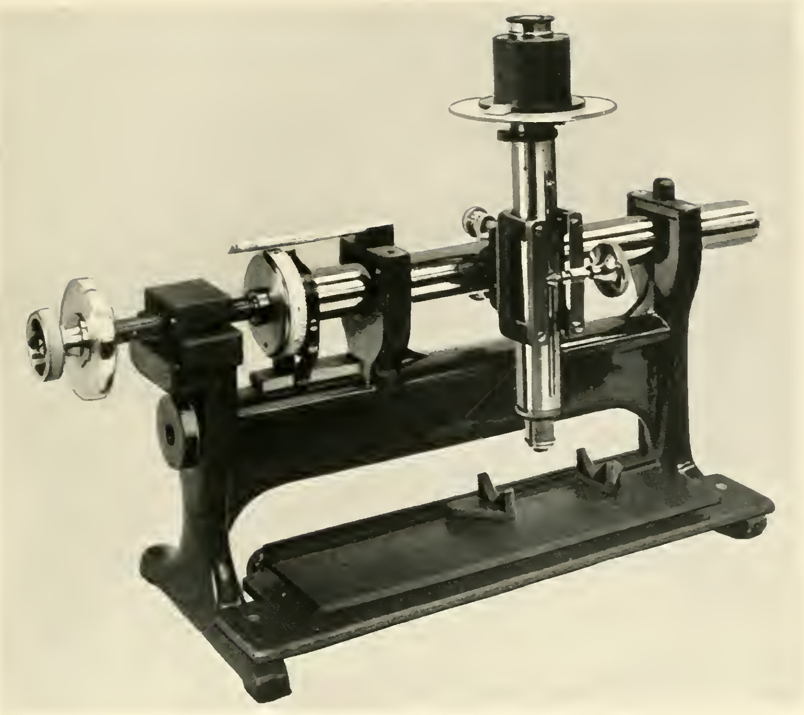


Fig. 17.—Cambridge Measuring Microscope

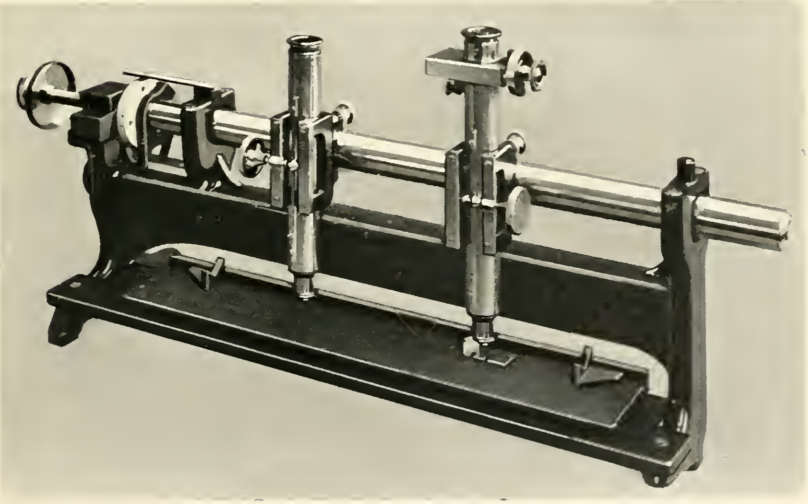


Fig. 27.—Comparator (Cambridge)

of the rod. The rod is thus free to rotate about its own axis. It may now be imagined that the surface of the rod is provided with an external screw thread, and that the five studs are provided with sections of the corresponding female thread; the surfaces in contact in each case are of considerable area, but the five different sections of the female thread (which thus form the " nut ") are entirely distinct and separated. The top section, in which the thread is of fair length, keeps the screw firmly pressed into the other four shorter sections. This tends to make the rotation somewhat stiff as compared with an ordinary nut, and good lubrication is necessary. The tension exerted by the spring is very considerable, and there is little if any backlash, but the wear on the screw may be somewhat increased. The purpose of the provision of the small cylindrical link* between screw and cylinder is to prevent the strain that would arise through any non-alignment of the axis of screw and cylinder were they fixed together. For the focusing of the microscope Lucas's patent slow-motion mechanism is used, a small double-flanged wheel moving a cylindrical rod by friction. It gives a smoother movement than the ordinary rack and pinion, and is entirely free from backlash.

For measurement of the angles of screw threads a special eyepiece provided with an angular scale is obtainable. The general form of the instrument lends itself to a number of uses, including employment as a comparator and cathetometer. The range is somewhat short, being 4 cm. only. Readings may be taken by estimation to 0·001 mm.

Photo-measuring Micrometer.—It is of course possible to keep the microscope fixed while the object moves, as is done in the ordinary microscope fitted with a mechanical stage and scales and verniers. The instrument by Messrs. Adam Hilger, Ltd., shown in fig. 19 embodies this principle on a much larger scale, and is intended for the measurement of astronomical photographs.

Screws of extreme accuracy are employed to traverse the stage in two directions at right angles, and in this case the sliding parts are extremely broad.

Points in the General Use of Measuring Microscopes.—In focusing† the instrument the first operation is the adjustment of the eyepiece to view the cross wires. An observer who has any astigmatic defect in his eyes should wear spectacles or, better, use an auxiliary correcting lens clamped at the correct angle over the eye lens of the instrument. Both eyes are opened and the accommodation relaxed. The eyepiece is then adjusted until the cross wires come clearly into view, while the accommodation remains relaxed as before. The microscope is then focused until image and cross wires are seen equally distinctly and without parallax.

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* See also p. 24.

† Many points in these remarks apply to a wide variety of instrument.

If a scale or similar object is being measured with a micrometer screw instrument and great accuracy is desired, the settings on successive divisions are made with care, moving the divided head in one direction only when just approaching a division in order to guard against backlash. Settings may be repeated, but care in taking up all backlash must be exercised. After the first set of readings all settings are repeated, moving in the opposite direction. The scale or object is then reversed on the stage and two sets of readings are taken in opposite directions as before. The mean values of the intervals are taken. Reversal of the direction of setting tends to eliminate in the mean result the effects of unequal wear in different parts of the screw thread in an instrument with backlash.

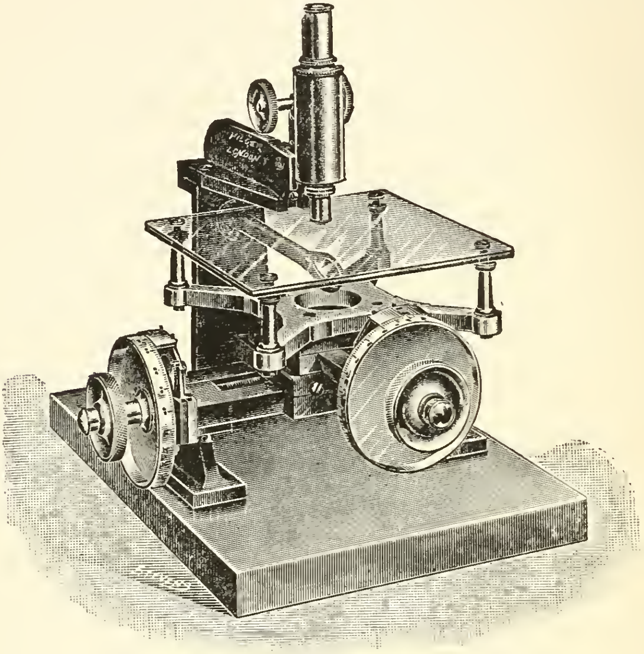

Fig. 19.—Photo-measuring Micrometer

In the case of unequal wear, a definite number of turns in one direction carries the nut through a greater distance than would be the case in the absence of wear; in the opposite direction the distance is less.

Reversal of the plate or scale tends to eliminate personal errors and errors due to singular characteristics of particular lines.

As in the majority of precise measurements, the question of temperature, involving expansions of materials and apparatus, is an important one and has to be remembered (p. 42).

CHAPTER III

Micrometer Microscopes: Comparators

The Micrometer Eyepiece.—In addition to the " micrometer screw " and " scale and vernier " methods of measuring displacements, the relative motion of a microscope and stage (or alidade and divided circle) may be very accurately found by a combination of an accurate and finely divided scale on the one part, observed by a microscope (with a micrometer eyepiece) on the other part. In this case the cross lines in the microscope are mounted on a frame which is moved by means of a micrometer screw.

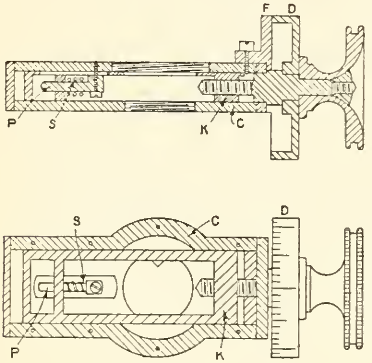

Fig. 20.—Construction of Micrometer Eyepiece

F, Fiducial pointer. D, Micrometer drum. C, Outer casing. K, Movable carriage. S, Spring. P, Plunger.

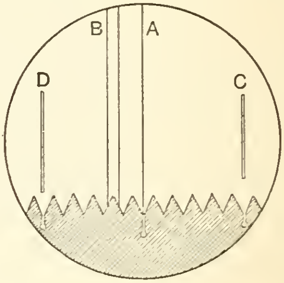

Fig. 21

Fig. 20 suggests the mechanical construction of the eyepiece and fig. 21 shows a part only of a typical field of view. A is a fixed central thread and B is the pair of movable parallel threads. A complete turn of the screw carries the movable thread from one tooth of the fixed " comb " to the next, so that " whole turns " are registered by the comb and fractions of a turn are registered on the micrometer head. C and D are images of the divisions of the standard scale. The use of the screw is to estimate the relative position of the fixed cross thread in the eyepiece to the images of the two scale divisions between which it appears. This estimation can evidently be performed by setting on C, A, and D in turn and computing the ratio of CA to CD. The numbers of the scale divisions are imagined to be increasing from right to left in the field of view. By selecting a suitable microscope objective and a correct tube-length for the microscope it is possible to arrange that ten revolutions of the screw shall carry the movable thread exactly from one image to the next. In that case the reading is simplified. It would thus be possible to read to a ten-thousandth part of the interval between the images if the micrometer head were divided into 100 parts, tenths of which could be estimated. With small circles it is frequently arranged that one whole drum revolution corresponds to the interval CD, when the reading is more simple still; further, the drum may be made to read in any convenient units. The drum should read zero when B is central, and readings will need one observation only. In the majority of cases the graduated drum is just friction-tight on its axis, so that it may be set to read zero when the movable cross wire is central.

If it is found that slightly more or less revolution than (say) exactly ten complete turns is necessary to traverse the threads from one image to the next, a small proportional correction, the correction for " error of runs ", has to be made to intermediate readings. This error may usually be eliminated by careful adjustments of the working distance and focusing of the microscopes, which usually are adjustable for this purpose.

Messrs. E. and R. Watts give the following note on parallax:

" The focus of the eyepiece and the image formed by the object glass must coincide on the plane of the cross hairs. If they do not, the hairs will appear to move over the object when the eye is gently moved up or down when looking at the object. This movement is called parallax.

" If, when the eye is moved to the left or right, no movement of the graduations relative to the wires is apparent, there is no parallax. But, if the graduation appears to move with the eye, the whole microscope must be moved towards the graduations, while, if the graduation appears to move in a contrary direction to the eye, the microscope must be moved away from the graduated circle.

" This movement is obtained by slightly loosening the clamp screws and gently sliding the micrometer the required amount up or down (probably a very small amount) and then tightening the clamp screws again."

It is evident that the possibility of finding a reading in this way is dependent on the absence of distortion in the image. We may consider that the true reading would be given by the relative position on the scale itself of the real image of the crossline A, formed by the microscope objective. If distortion is not present (to any serious extent) all dimensions in the image will be sufficiently nearly proportional to those in the object. Distortion may be neglected with simple microscope objectives of small aperture.

Micrometer microscopes are invariably employed in conjunction with specially divided scales when the most accurate measurements of linear or angular displacements are required.

It is easy to find the actual linear interval corresponding on the scale to one complete revolution of the screw, and it is then possible to measure the dimensions of small objects which can be placed sufficiently nearly in the same plane as the original standard scale. Such magnification methods have, however, to be employed with reserve.

The introduction of a micrometer screw brings into consideration all the various effects which were discussed in the last chapter, and which have again to be considered for work with any pretensions to accuracy. Measurements in which micrometer microscopes are concerned usually aim at a much greater *absolute* precision than is possible when the whole microscope or stage is moved by the screw; consequently the possibility of periodic errors in the micrometers has to be considered.

Gray* calls attention to several points of interest. Temperature changes may cause variations in magnification by thermal expansion of the tube, the screw, or the microscope lenses. A paper by F. L. O. Wadsworth† is (according to Gray) suggestive in this connection. " It points out that when an ordinary flint-and-crown objective is mounted in a brass tube the thermal expansion of the latter will almost compensate for the focal length change of the former." It is said to be " a common practice in astronomical observatories to mount the objective so that its distance from the reticule (system of cross lines) can be varied sufficiently to permit adjustment of the magnification from time to time ".

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* *The Optician*, 19th July, 2nd Aug., 1918.

† *Roy. Ast. Soc., Monthly Notices*, 1902, pp. 573, 591.

Such semi-uncertain changes of magnification are best investigated and, in the case of a micrometer microscope, allowed for by frequent measurements on some known length, but, in order to make a proper use of the measurements, something must be known of the errors of the screw. In a footnote of considerable importance Gray calls attention to the serious trouble found by Cornu,* and by Benoît and Guillaume,† caused by inaccurate focusing. The latter workers found that not only is accurate focusing necessary, but the use of objectives suffering from spherical aberration and coma may lead to systematic differences of nearly 4 μ in comparing length standards by different methods. Other possible causes of inaccuracy beside those previously mentioned are: looseness or insufficient rigidity in the framework on which the microscopes are mounted, defects in screw and slides, something rubbing against one of the spider lines, or a loose jewel in a bearing. Christie‡ and Gill§ have investigated the effects of wear on the periodic errors of screws. In the case of a gun-metal screw in brass bearings with errors at first negligible, it was found that four years' wear was sufficient to cause discrepancies of from 0·02 to 0·09 of a turn in measuring the same length (about 4·8 turns) with different parts of the screw. The errors of steel screws in brass bearings were not found to increase to such an extent.

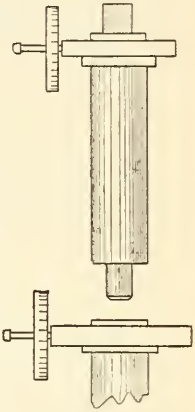
Fig. 22

In order to determine the errors of the screw, it is possible to employ a screw micrometer microscope (fig. 22), which is used directly to observe the spider lines of the first when the ocular of the latter has been removed. The auxiliary microscope should be capable of being moved as a whole, and the successive displacement of $\frac{1}{10}$ or $\frac{1}{5}$ of a turn of the screw of the first microscope are measured *always over the same part* of the scale of the auxiliary. Each interval may then be compared with the average length of the same intervals, and this average length is itself obtained in correct units by a determination of the magnification of the first microscope. Curves to exhibit the corrections throughout the whole range of the screw may be plotted; Gray shows how to apply such corrections to microscopes used in a comparator.

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* Cornu and Benoît, "Détermination de l'étalon provisoire international", *Trav. et Mém.*, 1894, p. 9.

† Benoît and Guillaume, "Mètres à bouts", *Trav. et Mém.*, 1902, p. 85.

‡ Christie, *Roy. Ast. Soc.*, *Monthly Notices*, 1876, pp. 18, 22.

§ Gill, *Roy. Ast. Soc.*, *Monthly Notices*, 1898, pp. 73–6.

Magnification Methods.—As was said in a previous paragraph, these methods of length measurement have to be employed with some care and reserve. A common method of measuring small objects is to mount a scale (divided or photographed) in the focal plane of a microscope, when the size of an object can be calculated (knowing the magnification) from the observed size of the image. The chief cause of trouble is the fact that a change of magnification, owing to an alteration of the focusing point, can sometimes be detected, while the corresponding loss of focus remains unnoticed. Some arrangement must therefore be provided to bring the test object and standard into the *same position* one after the other, and to make the method a purely comparative one.

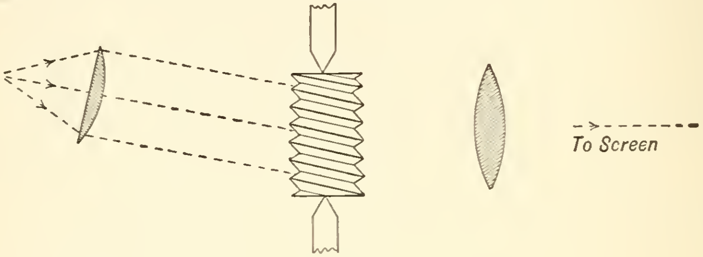

Fig. 23

A method of overcoming the difficulty (to some extent at least) has already been mentioned, viz. the use of a telecentric stop in the back focal plane of the objective. For many purposes, however, the loss of definition in the image is a somewhat serious drawback, and although the centre even of a diffuse line or image can generally be estimated with fair accuracy the use of the stop will be limited.

Comparatively recently an optical projection apparatus has been employed at the National Physical Laboratory in the department of metrology for checking certain standards, gauges, and screw-gauges. Fig. 23 shows a diagram of the apparatus as applied to screws. It will be realized that the main difficulties will be those already mentioned, viz. distortions and the correct location of the objects to be projected. Suitable lenses have now been computed for the purpose, but in the early experiments a combination of a photographic objective with a " binocular " objective formed a suitable projector, and the distortion in the lens combination could be experimentally minimized by varying the separation of the components. The light has to be accurately collimated before reaching the test object, and in testing screws the rays must be parallel to the rake of the thread. It was found possible to work to about $\frac{1}{1000}$ in. with a magnification of 50. The projection lens should have a focal length of between 2 in. and 4 in.

Standards of Length.—It will be necessary to summarize, in the briefest possible way, some of the main points in connection with scales and standards of length so that the necessity for and methods of using comparators may be correctly understood. It is hardly possible usefully to separate the optical and mechanical aspects of the question.

The original standards of length are purely arbitrary, and depend on reference to some one particular standard of length kept by the Government. Naturally a number of copies, more or less exact, are always made. These have to be checked by comparator methods against the original, and when altered so as to be practically exact they become sub-standards.

The standard of length is usually defined as the distance between two fine lines engraved on a bar, and these lines are often ruled in gold or platinum plugs to obviate tarnishing. Such standards are known as *line standards*. Other standards of length, no less useful, employ the distance between two flat or curved ends of a rod. The ends may be parts of a sphere struck with the mid-point of the rod as centre, to prevent errors through slight angular displacements of the rod. Such are known as *end standards*.

The subsidiary units of length are always derived as fractional parts of the whole. We shall see how *the error of graduation of a few nearly equal intervals may be detected by reference to the mean value of all of them*. Therefore, if some means of accurately estimating small differences of length is available, we can first make the subdivisions by trial and then correct each one by reference to the mean.

Having once obtained a " parent " divided scale of such a nature, it may be copied very easily by means of a suitable machine employing optical control. The ancient method was by means of the beam compass, but this was of course found to be inadequate when accuracy began to be required.

Dividing Engines.—Modern dividing engines usually possess a mechanism which, when set in motion, rules a line automatically at any required point. It would be possible to set up an arrangement consisting of a carriage moving on suitable slides between the standard scale and the scale to be divided. The carriage would bear on one side the automatic dividing mechanism which holds the tool ready to mark the undivided scale, and, on the other side, a microscope to observe the standard scale. The carriage could be caused to move so that images of successive divisions of the standard fall in turn on the cross-wire intersection in the field of the observing microscope, and at each such coincidence the dividing mechanism would be caused to rule a line.

Such a procedure would be far too slow for the production of the great number of accurate scales needed in scientific and engineering work, and in practice the dividing tool is moved by the automatically controlled rotation of a very carefully made micrometer screw. The " nut " is thus carried through a definite linear interval between the marking of each line. If a scale divided in this way is very carefully examined its errors may be found. The errors of the screw can then be inferred and compensated to some extent.

To explain the method of compensation, a simple arrangement may be pictured consisting of a screw passing through a fixed nut and pushing such a carriage as mentioned above. Suppose that a portion of the screw in which the pitch is a trifle too great passes through the nut, then the motion of the carriage would be too great for the corresponding angular turn of the screw. Imagine, however, that as this part of the screw is reached the nut is allowed to rotate slightly in the same direction as the screw. The result is that nut and screw rotate together, and to this extent the motion of translation of the screw is lessened. Simple means are provided, then, automatically to control the nut for the various parts of the screw, and the compensation is thus effected. The principle is of wide application and is used in the more accurate lathes intended for cutting standard screws, as well as in dividing engines.

To quote a description by the Société Génevoise: " The machine is driven automatically, that is all the work of dividing is done mechanically. Thus we avoid not only the errors due to the temperature of the body of the operator but also those resulting from his indifference or fatigue. The machinery offers the advantage of being more regular since its action is always constant. A special contrivance permits the correction of errors due to changes of temperature, and thus a division can be made at one temperature which will be correct at another given temperature. By the same means any kind of division of length can be made, even when the pitch of the main screw to the length of the division required should be irrational. The error curve of the screw has been conscientiously studied and corrected according to numerous tests made at the International Bureau of Weights and Measures. The error between two successive divisions has been reduced to less than $\frac{1}{1000}$ mm. on the whole length of the metre."

Comparators.—The principle of the comparator will be understood by reference to fig. 24. A and B are two microscopes. A has simple cross lines in its focal plane, while B possesses a micrometer eyepiece. C is a standard scale, and D is a scale on which the two marks shown are to be compared with the corresponding ones on C.

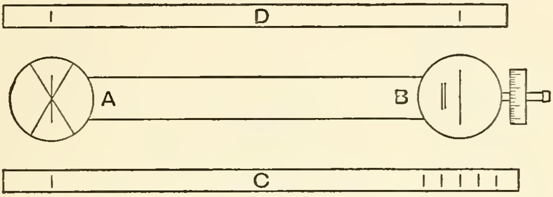

Fig. 24.—Comparator (diagrammatic)

In the first place the scale C is brought under the microscopes and the scale itself moved and adjusted till the image of the mark under A is in perfect focus and accurately on the intersection of the cross wires, and the image of the mark under B is also in perfect focus in the field of view. If a few graduations are visible in the field of B, the micrometer scale may be calibrated, that is, the linear interval on the scale corresponding to one complete turn of the micrometer head may be found. The cross wires in B are now made to coincide with the image of the proper division and the micrometer reading is taken.

Now A and B are fixed very rigidly, so that the distance between them may be considered invariable. The scale C is removed from under the microscope and D is substituted. As before, the images of the marks or lines to be observed have to be perfectly focused by movement or adjustment *of the scale*, and the image of the mark under A is adjusted to perfect coincidence with the intersection of the cross wires. If then the micrometer head has to be rotated before the coincidence is again obtained in B, the amount of the rotation is a measure of the *difference* in length between the pairs of marks on C and D.

The above description gives the essential principles in the use of comparators, and it is to be noted that the comparator or "substitution" principle is one of wide application. The order of accuracy which can be secured in the detection of small *differences* in dimensions (linear or angular) is usually far higher than is possible for the absolute measurement of any given dimension.

In practice the comparator is somewhat more elaborate than the simple instrument imagined for the above description. Each microscope may be furnished with a micrometer eyepiece. Special mechanical arrangements are provided for levelling the scales and for bringing them alternatively under the microscopes. The method of support of the scales has to be carefully studied lest bending strains should affect the linear dimensions. Usually the bars or scales are supported at the points mathematically determined as the best by Airy. They rest on a girder framework which runs on rollers and can be moved in a direction perpendicular to the length of the bars. Temperature control is of the highest importance. If the distance between the microscopes varies, the amount of variation must be found from careful temperature determination during the progress of any measurements. The scales themselves are usually immersed in water or paraffin and surrounded almost entirely by "water jackets" or "ice jackets"; a microscope observes a scale through a small aperture in such a jacket, the illumination of the scale being most conveniently secured by means of a 4-volt lamp with a disc of opal glass and a "vertical illuminator", the action of which will be understood by reference to fig. 25. The illumination has to be adjusted once for all before a set of measurements, for (as is the case in the use of the vernier) the adjustment of the illumination has a considerable influence on the readings. Fig. 26 shows a comparator of a large type such as is employed for the comparison of standards.

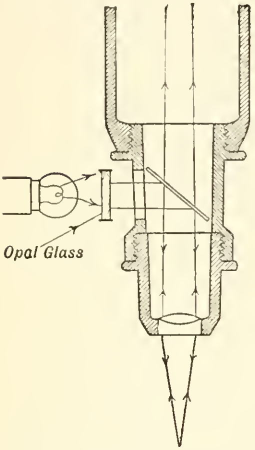

Fig. 25.—Vertical Illuminator

Comparator methods are of frequent application in the laboratory; take, for example, the exact checking of some short length (say one centimetre) on which it is not possible to use two microscopes, one at each end, in the manner described in the preceding section. In such a case a number of methods, all involving the substitution principle, might be devised.

1. We may use a micrometer microscope in which the whole microscope or table is moved and make comparative measurements on the test length against a standard, using identical parts of the screw thread of the instrument. In such a case the periodic errors of the micrometer screw would be the only ones likely to influence the comparison, and, even then, only if the difference involves a considerable fraction of a complete turn.

2. An alternative plan is to use two fixed microscopes to view both test and standard respectively. The latter are mounted on the same movable stage actuated by a micrometer screw. If the form of the slides is correct this method is very free from objection. Or again, both microscopes may be fixed to the slide as is shown in fig. 27, facing p. 28, which illustrates an obvious plan of turning the Cambridge measuring microscope into a comparator by supplying an additional microscope which can be clamped to any convenient part of the cylinder. For this purpose specially long cylinders are employed. It may be an advantage in a case of this kind if the microscope viewing the standard scale is provided with an eyepiece micrometer, but it would have to be used with a very light touch.

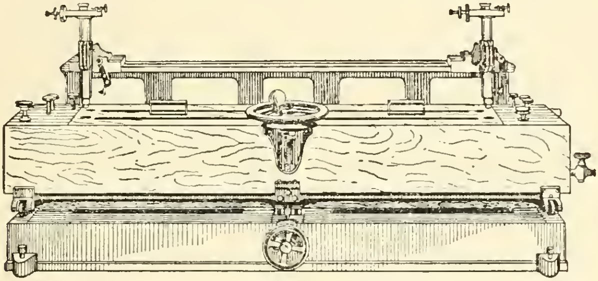

Fig. 26.—Large Comparator

3. If any considerable amount of work has to be done on checking intervals of the order of say 1 cm., it might be advisable to follow the plan sometimes used in calibrating the 1° divisions of large divided circles, i.e. the use of a specially constructed microscope with a double objective, and cross wires at about the same distance apart

as the divisions to be calibrated. A cross thread actuated by a micrometer screw is also provided. Test length and standard would be brought successively under the instrument.

It will be appreciated that in this case the setting will be made always very near to the optical axis of the image-forming objective. However, even in these measurements the greatest care in focusing adjustments is necessary.

Effects of Temperature.—A few words may be added with regard to effects of temperature on measurements of this class. The temperature of most buildings shows a fairly regular periodic change, and, in addition, any sudden changes of outdoor temperature generally cause corresponding (though lagging) internal changes. Rooms of ordinary temperature (60° F. say) are considerably below the temperature of the body, which is constantly giving off moist warm air by the breath and through the clothes. This may cause a perceptible rise of temperature in any parts of an instrument near an observer. Take for example the case of a divided circle with a mount of brass. Let the circle be 10 in. in radius, and suppose that a portion of the circumference 5 in. in length is raised in temperature by say 1° C. Assume that the radius of the circle does not change and that the coefficient of expansion of the brass is 0·000019, also that the whole expansion is operative in producing an increase in length of the part of the circumference considered.

It can then be easily deduced that the increase in angular measure of the angle represented by the above arc is 0·00001 approximately, or 1×10^{-5}. The angular measure of 1″ of arc $= 0{\cdot}5 \times 10^{-5}$. Therefore the increase of angle would be about 2″ in 30°. This treatment may suggest merely the possible magnitude of such effects.

In order to get over difficulties of this kind the instruments may be placed in a constant temperature room where the temperature is kept constant (by means of a hot-water system with a suitable thermostat) at about 20° C. In this case the observer's body is no longer at a temperature greatly in excess of the surroundings, although the conditions are by no means ideal for physical comfort.

The discovery of "invar", with its extremely low coefficient of expansion, has rendered possible great improvements in the accuracy of some length-measuring instruments and scales in which it is employed. Invar is a nickel-steel alloy. Its expansion is lowest in the neighbourhood of the temperature of 28° C. (1 to 2 parts in a million per 1° C.).

The Measuring Machine: comparison of length between line standards and end standards.—It is not usually practicable to check the lengths of end standards directly by means of the comparator, and special mechanical means have therefore to be employed. The most elementary measuring machine is the vernier or micrometer caliper, the use and action of which will doubtless be already familiar. In seeking to obtain the most accurate results, it is necessary to ensure that the specimens under test are always gripped with equal pressure when held between the jaws of the instrument. For the utmost refinement mechanical means are employed to secure an exact repetition of the same pressure in every measurement, and although by an educated touch it is possible for a skilled operator to obtain great accuracy on the micrometer calipers without the mechanical arrangement—usually a ratchet which gives way at a more or less definite pressure—yet much more delicate indicators, which far surpass any " estimation " measures, are used with the larger measuring machines.

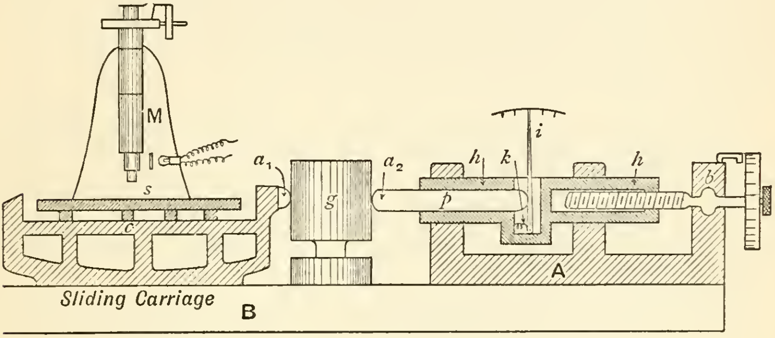

Fig. 28.—Société Génevoise Measuring Machine (diagrammatic)

In engineering measuring machines the most accurate micrometer screws with large divided heads (and frequently with special devices to correct the errors of the screw) are provided. In some of the best modern instruments the screw is supplemented with a standard linear scale and micrometer reading microscope, so that more accurate *absolute* measures of the standards or engineering parts may be obtained.

Fig. 28 shows diagrammatically the principle of construction of a measuring machine similar to that made by the Société Génevoise. The bed B carries on the right a micrometer headstock A, and on the left a sliding carriage *c*. Mounted on the latter is a standard scale *s* which thus moves underneath the observing micrometer microscope M which is rigidly fixed to the bed. In the Société Génevoise instrument displacements of the scale can be read to 0·00005 in. with the aid of the microscope. The micrometer headstock has a screw rotating in a bearing *b*, which thus causes displacements of a sliding member *h*. This member carries a plunger which is given a pressure to the left by the lever *i* and the spring *k*. Readings are always taken when the plunger is pushed inwards by a definite force (in practice about half a pound), this being indicated on a scale by the lever. To make an " absolute " measurement of, say, a cylindrical gauge against the standard scale, the carriage would be moved to the right till the " anvils " a_1 and a_2 come into contact. *c* could then be clamped and the pressure adjusted by the headstock screw, readings being noted both of this latter and of the micrometer microscope M.

The carriage *c* could then be moved to the left, the gauge would be placed in position, and the carriage moved up to the right again (by means of its fine adjustment screw) till the required pressure is obtained. The micrometer microscope can then be read again and the diameter of the gauge obtained from the difference of the two readings of M.

Having made one measurement of a gauge against the standard scale, the difference of various gauges can be obtained by the use of the micrometer headstock alone. The carriage *c* is kept clamped, the measurements being taken as with an ordinary micrometer caliper. In the Société Génevoise instrument readings with the screw can be made to 0·00001 in. Needless to say the actual construction and mechanical arrangements of the machine are more elaborate than those suggested in the diagram, which should, however, elucidate something of the underlying principles involved.

CHAPTER IV

The Divided Circle

Dividing Engines.—The divided circle is the essential feature of most angle-measuring instruments, and therefore the accuracy of its production is of fundamental importance in many kinds of scientific work. Also a certain knowledge of the properties of a circle and the apparatus used for taking readings on it is of the greatest consequence in securing accurate results from such instruments as theodolites. For a sketch of the development of dividing engines for circles the excellent articles (on " Graduation ") in the *Encyclopædia Britannica* or the *Encyclopædia Metropolitana* may be consulted.* Modern dividing engines are generally automatic. The circle being divided is rotated by a worm screw engaging with the specially cut teeth of a horizontal wheel, on the axis of which the circle is very accurately centred. By a special clutch arrangement the worm gear is moved through a definite angle controlled by the adjustment of certain stops which can be set for various angular intervals. At each stopping after a movement the engine draws a radial line by a special tool. The length of the lines can be automatically controlled so that, for example, every tenth division can be made longer than the others. A test of accurate adjustment is, of course, that the last mark on the circle shall coincide accurately with the first one after the circle has been completely graduated, but in practice no double marking would be allowed. It will be seen at once that the accuracy of the angle between divisions depends greatly on the accurate cutting of the teeth on the main wheel of the dividing engine. In a machine constructed by Messrs. E. R. Watts & Sons the main wheel was first hand-divided by trial till no single graduation erred by more than 0·6 sec. of arc; the teeth were then cut by reference to these divisions.

On completion of the graduation of the circle numbers are engraved against the main divisions. Verniers for the circle, if such are to be used, will then be graduated and the parts will then be removed to be mounted in the instrument. In the majority of cases the verniers in small instruments will be mounted so as to move with the telescope while the circle remains stationary. In larger instruments, however, such as the meridian transit circles of observatories, the micrometer microscopes, which take the place of verniers for reading the circle in accurate instruments, will often be fixed while the circle moves with the telescope. In small but accurate instruments for precision work in surveying, spectrometry, &c., the micrometers may move with the telescope, and the circle *may* be given an independent movement about its own centre so that readings may be taken over different intervals of the graduation in order to eliminate the effect of certain accidental errors of the divisions, thus avoiding the necessity of calibration.

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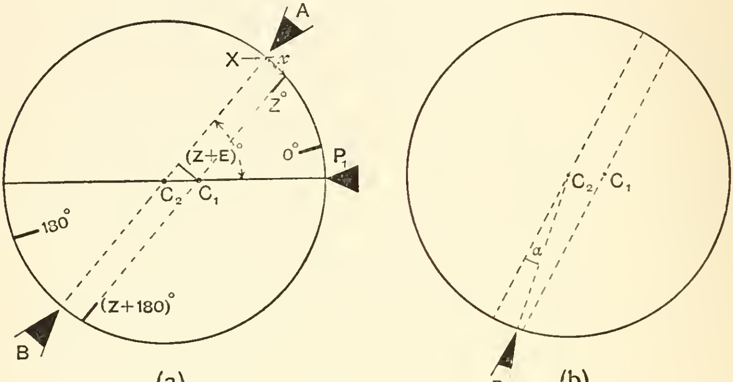
* See also the *Dictionary of Applied Physics* on " Divided Circles ".



(a) (b)

Fig. 29

It will be instructive to consider the case of the vertical transit circle with fixed micrometers, as certain of the possible errors will be clearly seen.

Suppose that the dividing engine has a perfect axis, and that the divisions are accurately made with respect to a point in a certain position relatively to the circular piece of metal on which the graduation marks are found. It will be the endeavour, of course, to mount the circle in the instrument so that its new centre of rotation C_2 coincides with the first " dividing " centre C_1 (fig. 29 (*a*)). Let it be supposed that these centres do *not* coincide, as suggested in the diagram. Four real graduations are indicated on the circle correctly made with respect to the centre C_1, viz. 0°, $z°$, 180°, and $(z + 180)°$.

In the case of rotation about the new centre C_2 fixed by the instrument, the dividing centre C_1 would rotate about C_2.

A pointer P_1 is shown at the extremity of the diameter passing through C_2C_1. Let the interval between this pointer and the 0° mark on the circle represent an angle E. Then the interval between the pointer P_1 and the $z°$ mark will be $z° + E$.

Draw a line through C_2 parallel to the line through $(z + 180)°$, C_1, and $z°$. Then the angle between this line and the diameter through C_2C_1 must be also $(z + E)°$ as indicated on the diagram.

If the circle be rotated (clockwise) about the centre C_2 through a true angle equal to $(z + E)°$, the point marked X in the diagram will clearly be brought under P_1. The nominal interval as indicated by the graduations passing under this pointer P_1 will thus be greater than the true interval by an amount indicated in the diagram by x. If the length C_2C_1 be measured in terms of the angular value e, which that length would represent when brought to the circumference of the circle, we have

$$x = e \sin(z + E).$$

Positional Error.—The above discrepancy between the indicated and actual values of the rotation would not exist if the divisions were all truly made with respect to the centre C_2. We may thus take it that those divisions at the extremity of the diameter through C_2C_1 are correctly placed, but that all others are generally wrongly situated. For example, the $z°$ mark ought to be placed at the point X. In other words the division $z°$ has a positional error of

$$x = e \sin(z + E).$$

The positional error is therefore periodic and obeys a " sine " law. A pointer *A* reading on the circle at the point X may in this way be considered to read wrongly by an amount equal to the positional error of the graduations in that region. Let the true or desirable reading be denoted by z', and let the actual reading be $z° + A$, where A will generally represent the fractional part of the reading, for eccentricity in well-made instruments is very small. Then since A is reading higher than its desirable indication,

$$z' = z + A - e \sin(z + E) \ldots\ldots\ldots\ldots (1)$$

The pointer *B*, situated at the other extremity of the diameter through X and C_2, will, however, be reading lower than its desirable value by an equal amount. Its true or desirable indication is $z' + 180°$, but it actually reads say $180° + z + B$; then

$$z' + 180° = 180° + z + B + e \sin(z + E). \ldots. (2)$$

It might easily occur, however, that the pointer *B* is not situated exactly at the extremity of the diameter mentioned. Let it be placed as indicated in fig. 29 (*b*), a little further round the circle, so that the line joining it to the centre C_2 makes an angle α with the diameter through X and C_2. Then as the true or desirable indication for the pointer is still $z' + 180°$, while its actual indication is now $z + 180° + B_0$, say, we have the relation:

$$z' + 180° = 180° + z + B_0 + e \sin(z + E) - \alpha. \,.\,. (3)$$

Difference in the Readings of Opposite Verniers.—Subtracting equation (3) from (1) we find:

$$A - B_0 = 2e \sin(z + E) + \alpha. \ldots\ldots\ldots (4)$$

It is thus possible to find the eccentricity *e* from a number of values of $A - B_0$ and z in a manner indicated in a subsequent paragraph.

Elimination of Eccentric Error in the Determination of Angular Intervals.—The difference of two "true" readings would clearly give the correct value of a corresponding angular interval. The method of deriving the true reading will be understood by adding (2) to (1), whence:

$$z' \text{ (the true reading)} = z + \frac{A + B}{2} \ldots\ldots (5)$$

This equation directs us to take the "degrees" from one vernier and the mean of the fractional parts *as defined above* in order to get the true or desirable reading.

When the pointers or verniers are not exactly at 180° apart with respect to the centre C_2, we obtain the corresponding equation by adding (3) to (1), whence:

$$z' = z + \frac{A + B_0}{2} - \frac{\alpha}{2},$$

or $$z' + \frac{\alpha}{2} = z + \frac{A + B_0}{2} \ldots\ldots\ldots (5a)$$

This equation applies to one particular setting of circle and pointers. For a fresh setting we might have a similar equation:

$$z_1' + \frac{\alpha}{2} = z_1 + \frac{C + D}{2} \ldots\ldots\ldots\ldots\ldots (5b)$$

Subtracting (5*b*) from (5*a*) we see that

$$z' - z_1' \text{ (the true angular interval)} = z + \frac{A + B_0}{2} - \left(z_1 + \frac{C + D}{2}\right).$$

In order to obtain correct measurements of angular intervals it suffices therefore to apply the rule for finding the " correct reading " as before, and subtract the two " correct readings ".

In order to illustrate the application of the rule, suppose the reading of one microscope were 12° 0′ 37″, and that of the opposite microscope 191° 59′ 58″. The first is taken as

$$z = 12°, \quad A = +(0' \, 37'').$$

The opposite reading is $z + 180° + B = 191° \, 59' \, 58''$, therefore

$$B = -(0' \, 2'') \quad \text{and} \quad \frac{A + B}{2} = \frac{35''}{2}.$$

The " reading " in this case would be taken as 12° 0′ 17·5″.

Sometimes three, four, or even six equidistant microscopes are provided to read the circle. In such a case it can be proved without difficulty that the effects of eccentricity are eliminated by taking the fractional parts of the readings from the mean of the indications of all microscopes in a manner similar to that already indicated.

Further Periodic Errors.—Now besides the effects of pure eccentricity we may have another periodic error due to a continuous motion (during rotation) of the centre of rotation C_2 with respect both to circle and pointers, this being due to any non-circular shape of the pivot resting in its V bearings. If the pivot has a uniform section of regular figure, such as an ellipse, it can easily be seen that the error caused in one microscope reading by such a displacement must be a periodic error. All displacements of the circle may be regarded as resolved into two displacements parallel and perpendicular (respectively) to the line joining two opposite microscopes or verniers. Displacements of the centre parallel to this line cause no change in readings, while displacements of the centre perpendicular to it cause equal and opposite changes. This effect " plays in " with the eccentricity in causing differences between opposite readings. If the figure of the pivot section is uniformly elliptical the motion of the instantaneous centre of rotation will repeat twice in one revolution, and Bessel has shown that the error of a *single* pointer due to this cause may be expressed as $\delta = f \sin(2z + F)$. If the section of the pivot is not uniform, however, the circle will not rotate in one plane.

Moving Verniers.—The whole of the arguments and mathematics applies equally well to the case of a fixed circle and movable verniers or microscopes. In this case C_1 is still the dividing centre which remains stationary, while C_2 is the centre of rotation of the mount of the verniers or microscopes called the *alidade*, with respect to which centre they maintain a constant relative angle.

The eccentricity now occurs between the dividing centre and the centre of rotation of the verniers. Errors due to the form of the pivots will now depend on the alidade pivots. The errors due to eccentricity and any elliptical form of the pivots, however, are completely eliminated by taking the mean of opposite readings as before.

In the case of instruments with vertical axes, any variations due to the form of the pivots may tend to be more uncertain, as the constraining force of gravity, in maintaining contact between the moving parts, is absent in certain cases; but conical bearings are generally employed.

Testing a Circle.—To find the eccentricity in either case from the difference of the " fractional parts " of readings of opposite verniers or microscopes, we may note that in taking a great number of readings round the circle and plotting such differences we shall have a curve represented (roughly perhaps) by the equation

$$A - B = a + 2e \sin(z + E),$$

if other effects due to ellipticity, &c., are small. In this equation E is a constant, $A - B$ represents the difference between the fractional parts of opposite vernier readings, and the other letters have the same significance as before.

In the diagram, fig. 30, we have, clearly, $a = 2'$, $2e = 2{\cdot}0'$, $E = 60^\circ$, and $A - B = 2' + 2' \sin(z + 60^\circ)$.

More Critical Tests.—In general, when we take into account the possibility of variations of greater frequency, similar to those due to the form of the pivots, the curve expressing the differences of the fractional parts of the pointer readings will be by no means so simple as the above. We may, however, find the *most likely* periodic law, and compare the observed differences with those indicated by the law in order to test the accuracy of graduation of the circle. The total correction for the periodic errors of the circle divisions themselves may be expressed as a function of the reading by a Fourier series:

$$\psi(z) = u' \sin(z + U') + u'' \sin(2z + U'') + u''' \sin(3z + U''') + \&c.$$

We may also make a distinction and express the ideal or desirable

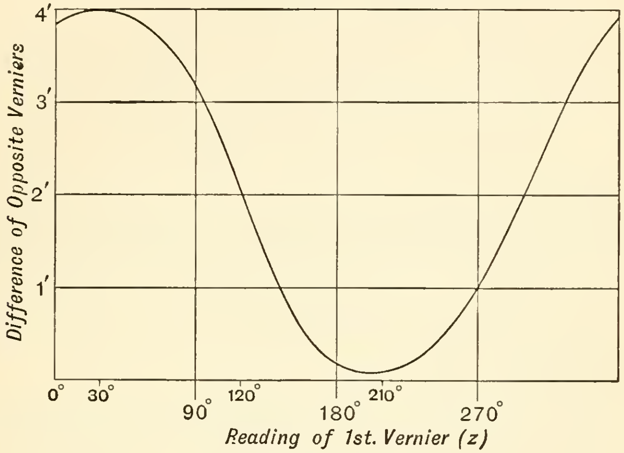


Fig. 30

reading of a microscope (i.e. the actual reading plus the correctional terms) in the form

$$z' = z + A + \psi(z) + \phi_0(z), \ldots\ldots\ldots\ldots (6)$$

in which $\phi_0(z)$ represents the part of the correction due to the form of the pivot, for the total apparent periodic error will be that due both to periodic errors involved in the graduation of the circle and to the form or action of the pivot.

For an opposite microscope, at an angular distance of 180° + a, we have:

$$z' + 180 = z + 180 + B - a + \psi(z + 180) - \phi_0(z) \ldots . (7)$$

(Note that the correction for the form of the pivots is equal and opposite.)

$$\therefore B - A = a + \psi(z) - \psi(z + 180) + 2\phi_0(z) \ldots . . (8)$$

Substituting in (8) the series for $\psi(z)$ given above, and also noting that

$$\psi(z+180) = u' \sin(z+180+\mathrm{U}') + u'' \sin(2z+360+\mathrm{U}'') + \&c.$$
$$= -u' \sin(z+\mathrm{U}') + u'' \sin(2z+\mathrm{U}'') + \&c.,$$

we find

$$\mathrm{B}-\mathrm{A} = \alpha + 2u' \sin(z+\mathrm{U}') + 2u''' \sin(3z+\mathrm{U}''') + \&c. + 2\phi_0(z).$$

We have already seen that for an elliptical form of the axis

$$\phi_0(z) = f \sin(2z+\mathrm{F}).$$

Let n be the difference of opposite vernier readings, then in this case

$$n = \mathrm{B}-\mathrm{A} = \alpha + 2u' \sin(z+\mathrm{U}')$$
$$+ 2f \sin(2z+\mathrm{F}) + 2u''' \sin(3z+\mathrm{U}''') + \&c. \ldots (9)$$

Calculation of the Constants.—To derive the values of the constants for a particular instrument it is necessary to go round the circle in steps of 5° or so, taking a series of readings giving the values of "B − A" corresponding to each "z". The constants in the series have then to be calculated by the familiar methods of harmonic analysis.

As a typical example of the testing of a circle in the manner described above, Chauvenet quotes the case of the meridian circle of the Naval Academy, U.S.A., in which it was found that the difference of opposite microscopes could be represented by the formula:

$$n = -13''{\cdot}23 + 10''{\cdot}18 \sin(z + 160^\circ\, 15')$$
$$+ 3''{\cdot}58 \sin(2z + 299^\circ\, 30') + 1''{\cdot}38 \sin(3z + 68^\circ\, 19').$$

The equation was obtained by taking a number of values of opposite vernier readings at a succession of places round the circle and calculating the coefficients of the successive terms in a manner similar to that just described.

A good idea of the relative accuracy of graduation of the circle can then be obtained by computing (from this formula expressing the generalized results derived from a digest of the indications of all the observed graduations) the values of the differences at the points of observation. A close agreement (within 1″ or so) would indicate a good accuracy of graduation. This, however, is only a test and not a calibration of the circle.

Use of Four Microscopes.—Four microscopes are often supplied to an instrument. All of these will be used in making an examination of the circle if possible, as greater accuracy is then attained. The method of using four microscopes for taking ordinary readings will be understood from the following treatment. Let A and B be microscopes at 0° and 180° as before, while C and D are at 90° and 270°. In the equations these letters represent the nominal microscope readings or rather their "minutes and seconds" parts. Adding equations (6) and (7) we find:

$$z' = z + \tfrac{1}{2}(\mathrm{A} + \mathrm{B}) - \frac{\alpha}{2} + \tfrac{1}{2}\{\psi(z) + \psi(z + 180)\},$$

or
$$z' = z + \tfrac{1}{2}(\mathrm{A} + \mathrm{B}) - \frac{\alpha}{2} + u'' \sin(2z + \mathrm{U}'') + u^{\mathrm{IV}} \sin(4z + \mathrm{U}^{\mathrm{IV}}) + \&c. \ldots\ldots\ldots\ldots (10)$$

The corresponding equations involving C and D are obtained by putting $z + 90$ in place of z, obtaining, if $(180 + \beta)$ be the angular distance of microscopes C and D,

$$z' = z + \tfrac{1}{2}(\mathrm{C} + \mathrm{D}) - \frac{\beta}{2} - u'' \sin(2z + \mathrm{U}'') + u^{\mathrm{IV}} \sin(4z + \mathrm{U}^{\mathrm{IV}}) + \&c. \ldots\ldots\ldots (10a)$$

On adding these equations we find:

$$z' = z + \tfrac{1}{4}(\mathrm{A} + \mathrm{B} + \mathrm{C} + \mathrm{D}) - \frac{\alpha + \beta}{4} + u^{\mathrm{IV}} \sin(4z + \mathrm{U}^{\mathrm{IV}}) \ldots (11)$$

The only term of the periodic series not eliminated is the one in $4z$. In fact, it may be proved that the only terms not eliminated by taking the mean of the readings of q microscopes are those terms involving qz and its multiples.

The difference of equations (10) and (10*a*) may obviously be employed to secure a series of equations of condition from which the coefficients of the terms in $2z$ and $6z$ could be found.

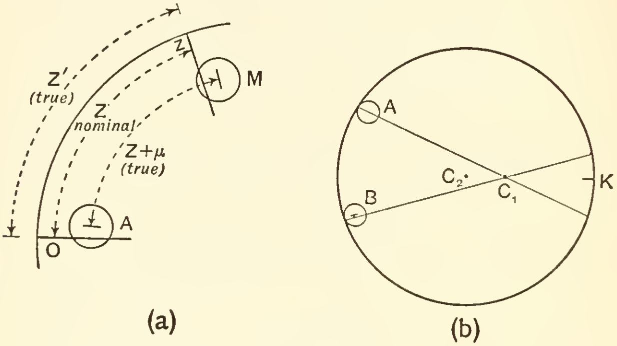
In the majority of cases the terms involving $4z$ and higher multiples are small in comparison with the accidental errors of graduation so that with four microscopes, or more, a correction for the accidental errors is all that is generally necessary, although particular cases might require special treatment.

Calibration of a Divided Circle.—It will be clear, then, that we now possess a method by which the most likely values, or the values most generally accurate, of the coefficients in a Fourier series expressing *periodic* errors may be found. It remains to determine the *actual total* errors of placing of *particular* graduations on the circle.

Fig. 31 (*a*) represents diagrammatically a part of the circle. Micrometer microscopes *A* and *M* are found nearly over divisions o and z respectively, say.

(1) Let the true angular distance between the graduations o and z be called z'.

(2) Let the true angular distance between the centre reference lines of the microscopes be called $z + \mu$.



(a) (b)

Fig. 31

(3) Let the "readings" of the microscopes be A and M respectively, that is to say, the fractional parts of the readings.

Then

$$z + \mu = z' + \mathrm{M} - \mathrm{A}.$$

It should be carefully noted that the true angular distance referred to above means the true angular distance with respect to the centre about which movement is to take place.

The difference between the actual and nominal values of the interval 0° to z° will then be

$$z' - z = \mu - (\mathrm{M} - \mathrm{A}).$$

If $\phi(0)$ and $\phi(z)$ denote the positional errors of divisions 0 and z respectively, then

$$\phi(z) - \phi(0) = z' - z = \mu - (\mathrm{M} - \mathrm{A}).$$

To find μ a repetition method is employed in which pairs of divisions Z and $Z + z$, $Z + z$ and $Z + 2z$, $Z + 2z$ and $Z + 3z$ are successively brought under the microscopes, giving a series of equations (in which $\phi(P)$ denotes the position error of the graduation P) as follows:

$$\begin{aligned} \phi(Z + z) - \phi(Z) &= \mu - (M' - A'), \\ \phi(Z + 2z) - \phi(Z + z) &= \mu - (M'' - A''), \\ \phi(Z + 3z) - \phi(Z + 2z) &= \mu - (M''' - A'''), \\ \cdot\quad\cdot\quad\cdot\quad\cdot\qquad &\qquad\cdot\quad\cdot\quad\cdot \\ \text{to}\qquad\qquad &\qquad\qquad\text{to} \\ \phi(Z + mz) - \phi(Z + \overline{m - 1}z) &= \mu - (M^m - A^m). \end{aligned}$$

Adding all these equations and dividing by m we obtain:

$$\frac{1}{m}[\phi(Z + mz) - \phi(Z)] = \mu - \frac{1}{m}\Sigma(M - A).$$

It is seen at once that if z is an aliquot part of the circumference $\left(\text{i.e.} = \frac{2\pi}{m}\right)$, $\phi(Z + mz) - \phi(Z) = 0$, for they refer to the same division. Therefore μ becomes known and may be at once applied to the successive equations to fix the value of, say,

$$\phi(Z + nz) - \phi(Z + \overline{n - 1}z),$$

where $Z + \overline{n - 1}z$ is a division so situated that its periodic error is zero. We may assume that the accidental error of this division is zero also. Then all other divisions will in general have positional errors with reference to it. The operator ϕ evidently includes both irregular and periodic errors. It will usually be sufficient if we assume that the division or graduation K—lying on the extremity of the diameter through C_2 and C_1 to the circumference, so that C_1 is between it and the rotating centre (fig. 31 (*b*))—is the division which has zero error. The interval between the graduations should appear shortest in this region owing to eccentricity. The division can be usually located more or less accurately from the observations of the differences of opposite verniers or microscopes. In fig. 31 (*b*) A and B represent two microscopes, C_2 is the centre around which the circle is turned (with imperfect form of pivots this will vary), and C_1 is the centre of the divisions or graduations. The apparent distance between two graduations (imagined in the figure as continued quite through the dividing centre) must be in the above case much greater in the position shown in the diagram than when the circle has been rotated through 180°.

Practical Determination of Apparent Positional Errors of the Graduations of a Circle.—To find the individual errors of a few divisions between two graduations of which the positional errors are known, it is necessary simply to use the required number of equations for "M − A". It will be seen that, as in the case of the calibration of linear scales, the values of each of say three divisions between two graduations of known positional error may be compared with the mean of all.

Therefore in an actual calibration the plan of working is to divide up the circle in a way which will be understood from the following example given for the calibration of the Greenwich transit circle.

1. Six equidistant main microscopes are provided. From these the total errors of each graduation bearing a number which is a multiple of 60° may be found.
2. Some additional microscopes at intervals of 20° are provided, which enables the intermediate divisions, 20° and 40°, between 0° and 60° to be dealt with, and so on all round the circle.
3. There are also microscopes at 25° apart. By means of these sections of 100° interval, such as between 0° and 100°, 20° and 120°, and so on, could be divided into four, and thus the errors of placing of all intermediate divisions such as 5°, 10°, 15°, &c., can be found by combination with the previous results.
4. In order to obtain the errors of placing of the 1° interval graduations it is now necessary to compare each 1° interval with the mean of five such intervals. A micrometer microscope with a double objective giving images of divisions 1° (nominal value) apart was employed for this purpose.

Calibration of any fractional parts of a degree will also be necessary. This may be carried out as detailed at the conclusion of the next section.

Separation of Irregular and Systematic Errors.—As was previously mentioned, the results found in the foregoing manner include both irregular and systematic errors. The series of tests exemplified by those on the Greenwich instrument yield finally the actual angles of the nominal intervals, and, by addition of the errors (see table on p. 60), the apparent positional errors. A table of these has to be made, taking the positional error of that division to be zero, which has zero periodic error, and correcting all other values with reference to this assumption.

We now refer to the original type of Fourier series expressing positional error:

$$\psi(z) = u' \sin(z + U') + u'' \sin(2z + U'') + u''' \sin(3z + U''') + \&c.$$

The results for the positional errors of the graduations at every 5° or 10° may be used in the manner already indicated to calculate the most likely values of the constants in the above expression up to, say, the third term if four microscopes are to be used for reading.

The expression thus derived is employed to calculate the *systematic* positional error of each division in the table above. The residual differences between the systematic errors and the total positional errors are taken to be the irregular or accidental errors.

Precautions for Different Types of Work.—It is well worth while to give a few simple tests even to the most ordinary divided circle. The vernier may be examined for length at about eight to ten places round the circle. Eccentricity may be tested for by taking opposite vernier readings at, say, every 20° and plotting the difference as suggested in fig. 30. Some care will be necessary in order to obtain intelligible results, and, unless the scale and vernier divisions are fine and properly illuminated, the eccentricity, if any is present, may easily be masked by more or less accidental differences. The amount and direction of eccentricity may be obtained from inspection of the curve, also the angle between the verniers.

It is as well always to take opposite vernier readings for *any* work (student's experiments or the like). No student should be encouraged to think that two verniers are provided on a spectrometer because one may sometimes get too close to the collimator to be conveniently readable. Careless habits of experiments are easily carried over into work demanding the greatest care. Students will probably find it simpler to take the separate indications of each vernier for any one angular interval and thus get the mean of the two angles.

Modern dividing engines usually work to a high order of accuracy, and the accidental errors of graduation in the majority of the circles with verniers found in ordinary theodolites, spectrometers, &c., are of an order which is not comparable with the possible accuracy of reading (5″ to 1′ of arc). In cases of this kind no calibration of the circle, &c., is called for, the only tests being those for the vernier (which will check the circle to some extent) and for eccentricity.

It was mentioned at the beginning of this chapter that in larger instruments where readings are required accurate to, say, 1″ of arc the circle may be given an independent movement. Any angular measurement may then be repeated over different regions of the circle and the effects of accidental errors of graduation are thus minimized. This plan was adopted in the ordnance survey of Great Britain.

For astronomical work with transit circles and the like, for spectrometry of the highest accuracy, and for any angular measurements where results are desired which shall be accurate to a fraction of a second, the calibration of the divided circle must be undertaken. The work is easier the larger the circle, but on the smaller circles, of diameter 18″ and thereabouts, the calibration is extremely difficult, and considerable training is necessary before an observer can obtain results consistent to a few tenths of a second. A few methods which have been found useful in cases of this kind may be worth description.

Methods for Calibration of the Circle in Laboratory Instruments.—The method previously described for the Greenwich transit circle involves the use of specially mounted micrometer microscopes, but these are not usually available in laboratory instruments which may possess two, four, or six equidistant microscopes and a " reader " for reading the figures on the scale. In one case, it was found necessary to test the divided circle of an instrument in order to determine the accidental errors of graduation.

In this case the four microscopes moved with the telescope, and the circle was ordinarily fixed although it could be given an independent rotation. Each degree was divided into six parts of 10′.

A beginning was made by comparing the subdivisions of each degree by means of simple micrometer settings on the limiting lines, the greatest care being taken to use exactly the same parts of the screw for each interval; the successive intervals were brought under the microscope by rotating the circle.

The necessity for tests of this kind was illustrated by the discovery of a well-marked periodic error (involving errors of more than 1″ of arc in these intermediate divisions) which repeated once in every such 1° interval round the circle. The sum of the subdivisions of 1° showed only a small variation of far greater period, thus showing that the " degree " graduations were comparatively accurate.

In order to check this result it is necessary to provide some means by which a microscope may be given small exact displacements in which the errors of setting are not greater than, say, 0·2″ of arc. A high-power eyepiece with extremely fine parallel cross lines (ruled on glass and 0·005 cm. apart) was fitted to the telescope of the instrument, which viewed at some distance a frame carrying three parallel wires of extreme fineness backed by white opal glass and illuminated from behind. The angles subtended by the intervals between the wires corresponded to 10′ and to 1°. It was found possible, by exercising the greatest care, to make settings of the telescope, on one wire, which were consistent to the required accuracy. The distance of the frame can be adjusted to bring the angles subtended accurately to the required values.

In testing a 1° interval the telescope is directed to one wire, and the divided circle is moved by its fine adjustment screw so as to bring one circle division into a suitable position for observation by one of the reading microscopes. The reading is now taken exactly on the drum of the eyepiece micrometer screw. The telescope is then directed to the other wire, when the micrometer cross lines ought to be brought accurately to the next circle division. Any discrepancy, and thus the nominal value of the interval, is found by the micrometer (in seconds of arc). The 10″ intervals are similarly treated to verify the previous results.

The next step is to add the 360 nominal intervals as found, when any error in the adjustment of the subtense of the wires will be revealed as an excess over or deficiency from 360°. Let the result be $(360 + x)°$; then each 1° interval must be multiplied by $\frac{360}{360 + x}$ in order to reduce it to the proper value.

It would be well, in making a thorough test, to perform a similar experiment for 5° intervals and compare the results with those found by taking the sum of five successive 1° intervals at a time.

By means of the four microscopes on the instrument each 90° interval can, by rotating the circle under them, be compared with the mean of all four such intervals as described on p. 55. It is a useful check to compare the results with those obtained by adding the results as found by the telescope method for each of the four intervals of 90°.

Having found the successive intervals, the apparent positional errors of the graduation can be computed by the sum of the apparent errors, as is suggested in the table.

Interval (Nominal).	Apparent Dimensions.	Mark.	Apparent Positional Error
0°–1°	1° 0′ 1″·5	0°	0° (say)
1°–2°	1° 0′ 1″·3	1°	+ 0° 0′ 1″·5
2°–3°	1° 0′ 1″·1	2°	+ 0° 0′ 2″·8
3°–4°	1° 0′ 0″·9	3°	+ 0° 0′ 3″·9
&c.	&c.	&c.	&c.

The 0° graduation is assumed to be correctly placed in the above table, but the proper graduation must be selected for the purpose in each special case.

The remaining procedure has already been explained. The results in the fourth column are used for calculating the systematic errors, and the accidental error can then be separated.

It will be as well to keep clearly in mind the various types of error. The systematic errors involved in rotating the circle under the microscopes will be quite different from those found if the microscopes are rotated over the circle. If it is desired to investigate the systematic errors in this latter case, the telescope method might still be employed with a spectrometer if a specially good plane mirror, which can be mounted exactly over the centre of rotation of the prism table, is available. The image of the frame and wires would then be moved by rotation of the table and mirror into suitable position to be viewed by the telescope as it is moved in steps round the circle. Certain obvious difficulties would have to be overcome. The accidental errors of graduation, with respect to any one division, ought, however, to be identical no matter what the method of determination, but in practice this would depend on the completely successful elimination of the systematic errors.

CHAPTER V

Theodolites and Sextants

THE THEODOLITE

Historical.—The theodolite is one of the oldest instruments discussed in the present book, but it is undoubtedly derived from others of much greater antiquity. The astrolabe, probably invented by Hipparchus of Bithynia about the year 150 B.C., was intended to observe the altitude of the stars. It consisted essentially of a flat circular plate which could be suspended in a vertical plane by a shackle and ring on the edge. Moving over the plate and pivoted on a pin passing through the centre was the alidade which carried at each end vanes with open sights. In the Arabian schools the instrument was provided with a divided circle, and the " tablets " and " rete " were also added to facilitate the solution of astronomical problems. The instrument was, however, used for surveying as well as astronomy, and for the determination of angles in azimuth. The plate could be mounted in a horizontal plane. Verniers for reading the circle were added to the alidade during the seventeenth century. Until near the end of the sixteenth century no *surveying* instrument carried both horizontal and vertical circles, but in A.D. 1571 Leonard Digges constructed a " theodolitus " which consisted of a horizontal circle over which moved a vertical semicircle carrying open sights on its diameter. Angles between the sighting line and the horizontal were read on the semicircle by means of a plummet. Altazimuth instruments carrying both horizontal and vertical circles were, however, in use long before this time in astronomical observatories, and the development of the theodolite was thus guided. Telescopic sights, invented by Gascoigne in A.D. 1639, were not generally adopted for surveying instruments till the end of the seventeenth century, and bubble levels, the invention of Thévenot, did not replace the earlier plummet and gravity devices till the beginning of the eighteenth century.

Great advances were made in the latter half of the eighteenth century, when the use of dividing engines, originating from Hindley's machine for cutting the teeth in wheels for clockwork, made possible the production of accurate circles and paved the way for the work of Ramsden and Troughton. To both the latter we owe noteworthy improvements in the construction of theodolites, and especially the circles, while to Ramsden must also be credited the invention of the eyepiece bearing his name, the application of the eyepiece micrometer to a microscope, and the employment of the combination for reading the divided circle in place of the vernier.

From the early part of the nineteenth century and onwards beautiful examples of theodolites were constructed.

Development.—As indicated above, the theodolite is the descendant of the astrolabe and altazimuth instrument, and consists generally of a telescope mounted so that it is capable of rotation about horizontal and vertical axes, while the amount of rotation is measured on divided circles provided for the purpose. For some special purposes one of the circles may be omitted, usually the one with a horizontal axis, or there may be merely an approximate " angle indicator " for rotation in altitude.

In its modern form the theodolite is purely a surveying instrument, and many sizes are in use, from very small portable types with circles of 3-in. diameter to the larger instruments possessing 10-in. or larger circles read by micrometer microscopes and intended for standard surveys. Progress in the design of the smaller instruments has been concerned with securing greater portability, compactness, lightness, and the protection of the parts from dust and the weather. In some classes of instruments the optical system has been improved so as to permit of the more efficient use of the telescope for tacheometry. Broadly speaking, however, the above essential features of the construction are common to all.

Many points in connection with the use of divided circles have already been discussed (Chapter IV), and these are to be applied to all such angle-measuring instruments. The directions for focusing (p. 29) also apply to the present case.

Mechanical Construction.—The mechanical construction of the instrument has to be designed to ensure the geometrical accuracy of the performance of divided circle and alidade. Fig. 32 gives a general presentation of a small instrument, and in fig. 33 (*a*) the construction of a typical form of vertical axis is indicated; the clamps and fine adjustment fittings which are always provided for the moving

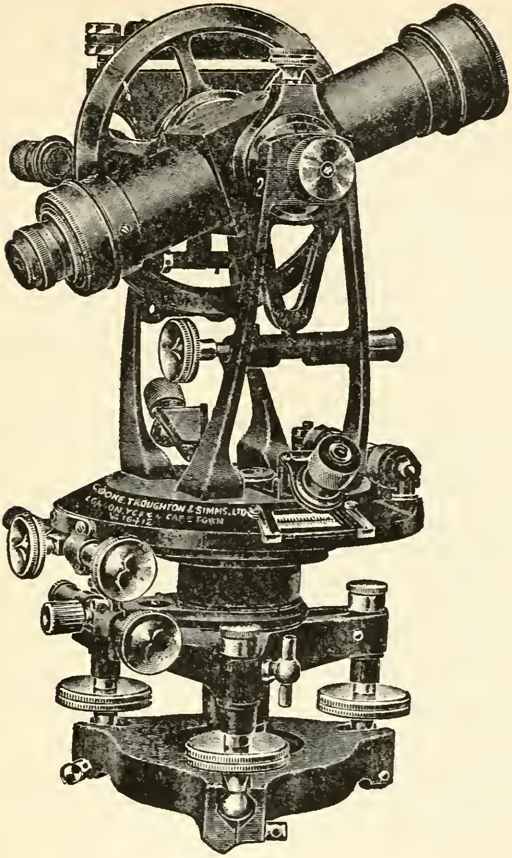
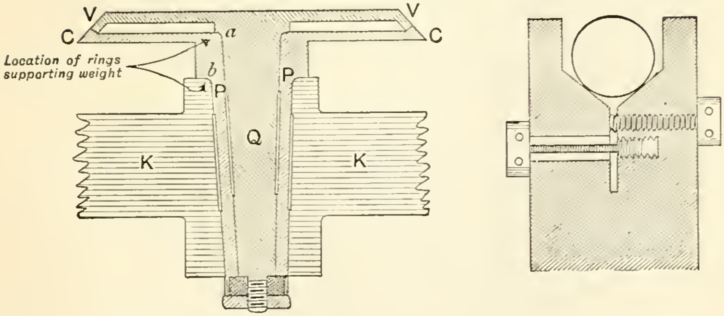


Fig 32.—Small Theodolite



(a) (b)

Fig. 33

parts are not shown. The circle C is carried by a hollow cone P (shown sectioned in the diagram). The exterior surface of this cone rests in the corresponding conical bearing turned in the casting K (the tribrach) which carries the foot screws. The alidade and telescope standards are carried by the interior solid cone Q bearing in the part P. It is usual to clear away a centre part of the exterior of a cone so that the contact is made in the upper and lower regions only. The main portion of the weight should be borne, not on the surfaces of the cones, but by the ring contacts suggested by *a* and *b* in the figure. The spindles are kept in place by the nut and screw at the lower end of Q. The angles of the cones are from 3° to 5°, and these parts serve mainly to regulate the geometrical truth of the rotation.

With the above arrangement not only can the verniers be moved relatively to the circle, but the circle can also be used in various positions for the same angular measurements (p. 46). Surfaces bearing against each other should have a different hardness for the best effects as regards smoothness of motion and even " wearing ". The part P could be made harder than K or Q. Metals commonly used for the bearing parts in older instruments were brass, gunmetal, bronze, &c. Phosphor bronze (or bell-metal, which is still harder) is now employed where great hardness is required. In some modern instruments a modified form of axis is being introduced (p. 73).

The plate or " table " of the instrument is carried, in small instruments, by the centre cone, and it is on the table that the supports for the telescope are fixed. In some cases, as in Messrs. Watts' instruments, the table and the supports are made in a single casting. These supports are usually of some " girder " type designed for lightness and strength, and they carry at the top the bearings for the telescope axis. In the simpler instruments these are often of the plain square V type, and the pivot, which has to be made as nearly circular as possible, rests in them under gravity. On at least one side the relative height of the bearing must be adjustable, and one method of securing this is indicated in fig. 33 (*b*). Push-and-pull screws increase or diminish the width of a saw-cut made in the mounting from the corner of the V, and thus permit the pivot to fall or rise through a very small interval. In a better construction both the whole V supports move in slides, and their height can be adjusted by capstan screws. In some modern instruments, especially those in which the telescope is not removed from its bearings when packing the instrument in its case, these bearings of the horizontal axis may be protected or covered in. Cylindrical bearings, if such are employed, can be made segmental in order to allow of slight adjustments.

The vertical circle is mounted coaxial with the pivot of the telescope and usually rotates with it. The alidade for this circle is usually given a short conical pivot; it carries a long and sensitive bubble but otherwise it has no weight to support. A clamp and fine adjustment permit it to be fixed or given small angular motions relatively to the supports of the telescope. The supports and tube of the telescope are made very strong in order to avoid any flexure under gravity, and the focusing arrangements have to be designed with this point in mind.

Spirit Levels.*—In the adjustment of the theodolite before use, the first thing is to make the central axis truly vertical. This is done by the use of spirit levels, which if constructed from long tubes with a very slight curvature may be made extraordinarily sensitive. The length of the bubble, the diameter of the tube, and the radius of curvature are the chief factors affecting the sensitivity. With a radius of curvature of 200 m. in the tube and a bubble of suitable length it is possible to obtain a displacement of 1 mm. per second of arc change in the tilt of the tube.

" Level " tubes were formerly selected specimens of suitably bent tubes, but modern ones are specially ground, with the help of a thick curved wire passing through the tube. Many modern levels are machine ground. Henrici† summarizes the requirements of good levels:

1. The bubble should move as quickly as possible.
2. Any tilt of the tube larger than the permissible error of adjustment must make the bubble move a noticeable amount. (The greater the radius of curvature the greater the movement.)
3. The curvature of the tube must be uniform.

Quickness of movement is secured by a long bubble with a short radius of curvature of the tube, so that quickness of movement can hardly be attained with great sensitiveness.

Spirit levels may be tested by the use of a suitable clinometer carrying a bar with adjustable V supports for the tube. The bar is hinged at one end and the level of the other end is altered in suitable steps* by means of a micrometer screw with accurate bearings. Where an instrument has to be used in extreme ranges of temperature, the length of ordinary bubbles and therefore the sensitiveness of the motion is liable to considerable variation, but Messrs. E. R. Watts & Sons, Ltd., have recently introduced a bubble which does not vary with temperature. Ordinary bubbles have circular tubes, and in this case the expansion of the liquid (usually benzol, xylol, chloroform, alcohol, or ether) is sufficient when the temperature rises to cause a diminution in the volume, and therefore in the length, of the bubble. The change of surface tension of the liquid also enters into the action. In Messrs. Watts' bubble the tube has an elliptical section, and in this case when the temperature rises the expansion of the liquid is taken up by a diminution in the cross-sectional area of the bubble, the length remaining constant.† The mounting of the bubble tube in its metal casing is also to be considered. The usual mounting in plaster of Paris is liable to produce strain and distortion with a consequent "stickiness" of the bubble. It is better mounted in cork-lined Y rests and held in position by springs. In any case the mounting should give play for the relative motion caused by the different expansion coefficients of metal and glass.

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* See also Chapter VI.

† *Trans. Opt. Soc.*, Vol. XX, No. 3.

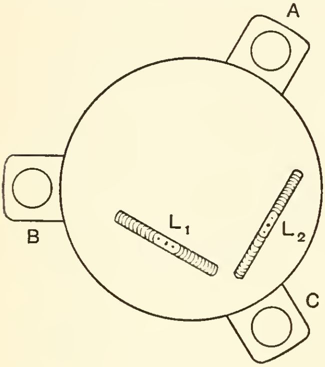

Fig. 34

Adjustment of the Theodolite.—(1) *The Vertical Axis.* Two spirit levels at right angles to each other are provided on the plate of the instrument, one can be placed parallel to the line joining two levelling screws. By reference to the diagram (fig. 34) it will be clear that motion of the screw A cannot greatly affect the adjustment of the level L_1.

The method employed is therefore to centre the bubble in L_1 by motion of screws B or C and to centre L_2 by means of the screw A. The adjustment should remain perfect when the plate has been rotated through 90°. If this condition cannot be attained after a few trials, the levels themselves require adjustment with regard to the vertical axis. For details of such adjustments, reference may be made to the description of the adjustment of the " level ", p. 91. The levels are adjustable by means of capstan screws at both ends.

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* Effects of "hysteresis" due to "stickiness" of the bubble should be looked for when reversing the direction of the clinometer movement.

† S. G. Starling, *Trans. Opt. Soc.*, XXIV, No. 5, 1923.

(2) *Collimation of the Telescope.*—The line joining the " principal point " of the objective to the central vertical cross wire may not be exactly at right angles to the horizontal axis (fig. 35 (*a*)). The instrument is levelled, and a particular image point falling on the vertical cross wire is noted. The instrument is then turned through exactly 180°, and the telescope is made to " transit ".* If the same image point does not fall on the cross line, half the error is corrected by means of the capstan screws provided for the adjustment of the frame holding the webs of the diaphragm, and the process is repeated.

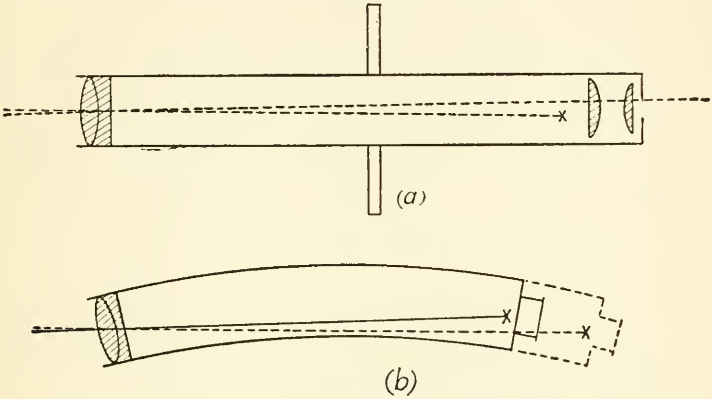

Fig. 35.—Collimation Error

(3) *The Horizontal Axis.*—The next step is to ensure that the telescope will transit, moving on its horizontal axis, in a truly vertical plane. This will not be the case unless the bearings of the telescope axis are properly levelled so as to make this horizontal axis perpendicular to the vertical axis. The method followed is to sight exactly two points or objects, one as far above the other as possible. The instrument is turned on the vertical axis through about 180°, and the telescope is made to transit in redirecting it to the higher point. If now on depressing the telescope, allowing no more rotation on the vertical axis, the same lower point is not reached, half the apparent error is corrected by adjustment of the horizontal axis and the process is repeated.

(4) *Adjustment of the Level on the Index Arm.*—The direction of

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* Transit—to turn through 180° on the horizontal axis.

the line of sight through the horizontal cross wire must be exactly horizontal when the level on the index arm has its bubble central and the verniers on the vertical circle are reading zero. If all the previous adjustments have been correctly made the remaining procedure is as follows:

The verniers on the vertical circle are made to read exactly zero, and the instrument is directed to a vertical surveying staff at as great a distance as possible. The reading on the staff is taken. The telescope is then made to transit on the horizontal axis *exactly* 180°, and by rotation on the vertical axis it is again directed to the staff, the reading on the staff being noted again. Half any error is corrected by moving the lever which controls the position of the index arm; a small adjusting screw for this purpose is provided. The operation is then repeated until the adjustment is perfect. When this is the case the level on the index arm has the bubble brought to the centre of its run by the use of the adjusting screws at the end.

Obliquity of Axes.—Referring to fig. 33 (*a*), it will be noticed that P rotates in quite a different axis from Q. Although the hollow cone P will always have been turned with the greatest care on an accurate lathe, it may chance that the axes of its interior and exterior cones are not quite truly in alignment. The test for this may be made by supporting the instrument very rigidly and directing the telescope to some distant object, preferably a staff. The alidade which bears the telescope is now prevented from rotating (thus securing that the inner cone Q does not rotate), while the circle clamp is loosened and the circle is given a rotation. This makes the hollow cone P rotate between K and Q; the result will be that if the axes are not co-linear Q will suffer a small angular movement in altitude which may be detected and measured by observations with the telescope. In well-made instruments the axes should be co-linear within 3″ to 5″ of arc.

In some theodolites the hollow cone is not employed; the inner axis is made longer than usual and serves as the common spindle around which the horizontal circle axis and the lower axis revolve. This device eliminates the possibility of obliquity of the axes.

Collimation Errors.—If the tube of a theodolite or level is curved, any adjustment of the focus by movement of a draw tube will cause an error in collimation. The effect is illustrated in fig. 35 (*b*).

The trouble is said to be overcome by the use of duralumin for the telescope tube and the adoption of an internal focusing device, by which the adjustment of the focus is secured by moving a negative lens (see fig. 39) inside the main tube, although it is clear that in the latter case the freedom from variation in the collimation will still be dependent on the precision of the movement of the negative lens. It is easier, however, to provide slides sufficiently accurate to prevent such collimation errors if the negative lens is used.

The collimation always varies with a change of focus unless the locus of the intersection of the cross wires (where these are moved), or the locus of the nodal point of the object glass if this is moved, remains in the original line of collimation. A method of testing is given in the next chapter.

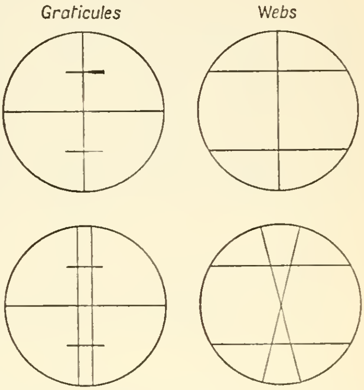


Fig. 36.—Simple Arrangements of Diaphragms

Diaphragms. — The cross-wire system varies considerably between various instruments. Lines ruled or obtained by photography on glass (graticules) are displacing spider web, which used almost exclusively to be employed for the " stadia " lines in the diaphragm. The exact spacing of sets of parallel lines is of great importance in tacheometry, and spider lines are somewhat difficult to set exactly, besides being liable to sag. Fig. 36 shows typical arrangements for the cross-wire system.

Tacheometers and Anallatic Systems.—In surveying, it is frequently necessary to determine the distance of a point to which a chain cannot conveniently be stretched. In such a case a graduated " staff " is employed, and is viewed by means of the telescope in a theodolite or level. Imagining that the staff is vertical, a certain number of its imaged divisions will appear between two horizontal cross lines in the field of the instrument, the number increasing with the distance of the staff. The diagram (fig. 37) will make the relation clear.

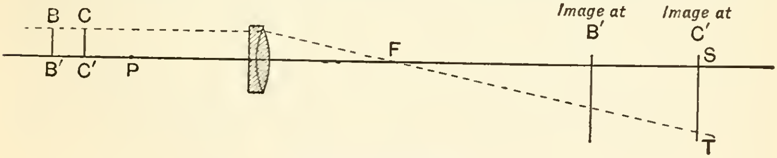


Fig. 37.—Tacheometry

We will imagine that these horizontal cross lines are perpendicular to the diagram at C and C′, the latter being on the axis of the telescope. This axis may be imagined to be prolonged till it meets the staff at S; it is seen, of course, that the point S will be imaged at C′.

According to the distance of the staff, the cross lines will have to be adjusted to coincidence with the position of the image (they are moved, of course, parallel to the axis). Our problem is now to find the point on the staff which will be imaged on the higher cross wire in its various positions. It will be sufficient to take a single ray to answer the question. If it passes through and helps to form the image it must come from the object. Such a one is the ray BCFT, which always passes through the top cross line, no matter what the distance of the latter from the lens. As it is parallel to the axis on the left of the object glass it evidently passes through the anterior principal focus F of the lens. Therefore the length of the staff imaged between the cross lines will be that found between its intersection with these two rays, which is proportional to the distance FS. Hence in using the tacheometer the length of the staff thus found by observation in the telescope will have to be multiplied by the appropriate constant f/e (when f is the focal length of the object glass and e is the vertical distance between the cross lines), and this will give, not the distance from S to the telescope object glass, but the distance of S from its anterior focus F. The position of this point with regard to the telescope is usually specified by the maker of the instrument. In many tacheometers the stadia lines are placed so that f/e is 100 exactly.

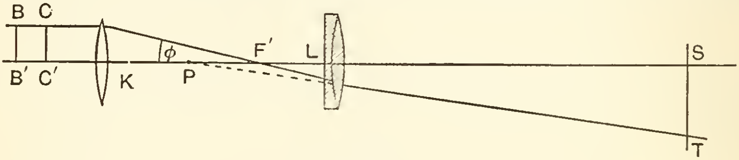


Fig. 38

A device introduced by Porro, however, enables the observer to find distances with respect to the centre of rotation of the telescope. This device is the anallatic lens. When an anallatic lens is used it is placed at K (fig. 38) between the webs and the object glass. Once again the points of the staff focused in the cross lines must lie on the track of the ray BC. This passes through the focus F′ of the lens K and is deflected by the object glass so that it appears to be derived from the point P on the axis. The lenses are so chosen and arranged that P is the axis of rotation of the instrument. Now the intercept of the staff image will evidently be proportional to the distance PS, which is the condition required.

In calculating the lens for a given object glass and to produce a certain constant r for the instrument, the position T can be found on the staff such that $SP = rST$. The location and direction of the ray TP becomes known, and also, by a straightforward calculation, its angle ϕ with the axis after passing through the lens. The focal length f_1 of the anallatic lens and the separation e of the cross wires have then to be chosen so that $e/f_1 = \tan\phi$. The anallatic lens has evidently to be placed so that its principal focus is at F′, F′ and P being conjugate points with respect to the object glass. This condition may be written:

$$\frac{1}{KL - f_1} - \frac{1}{PL} = \frac{1}{f},$$

or

$$KL = f_1 + \frac{f.PL}{f + PL},$$

which gives the required distance between the two lenses.

Internal Focusing by the Anallatic Lens.—In a patent by Carl Zeiss, 25,622 (1910), an anallatic lens is used to focus the instrument. It is arranged that the lens when set for distance shall be half-way between the diaphragm and the point F′. When focusing with the lens for nearer positions of the staff it will be seen that the lens must be moved nearer the object glass. The ray through the original point F′ now crosses the diaphragm farther from the axis, so that, in order to maintain the original value of the constant, the distance apart of the cross lines should be increased. However, it can be shown that, provided the anallatic lens is of such a power that a shift by half the distance from its original position to the point F′ is sufficient to adjust the focus, little error results from fixed diaphragm lines. The instrument will give accurate indications for all distances over 30 ft., and, having internal focusing, may be made air-tight. The rod actuating the rack and pinion for the focusing movement passes into the instrument through a stuffing box.

Internal Focusing with Negative Lens.—Fig. 39 shows a construction in which a short-focus negative lens is used for internal focusing. The instrument does not function quite correctly as a tacheometer, but the error is small enough to make it negligible for certain purposes. In Messrs. Casella's patent the focusing head is placed at one extremity of the transit axis in a tacheometer theodolite, where it is always accessible.

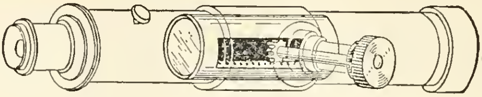


Fig. 39

The following table gives some results of actual tests made with such an instrument, in which actual measured distances are compared with those found from the tacheometer by reading on the staff.

Actual Distance.	Stadia Measure.	Error per Cent.
10 ft.	9·78 ft.	2·0
20 ,,	19·72 ,,	1·5
30 ,,	29·76 ,,	1·0
40 ,,	39·70 ,,	1·0
50 ,,	49·83 ,,	0·4
60 ,,	59·71 ,,	0·5
100 ,,	99·74 ,,	0·25

By using a constant to be added to the stadia measure it is clear that the above residual errors could have been considerably lessened. In the modern Zeiss levels it is the practice to give the formula expressing the requisite constant. Thus for model II the formula is $D = 0{\cdot}15\text{ m.} + 100l$, where D is the distance of the staff, and l is the intercept. Further details on this point will be found in a paper by Henrici, *Trans. Opt. Soc.*, XXII, No. 1.

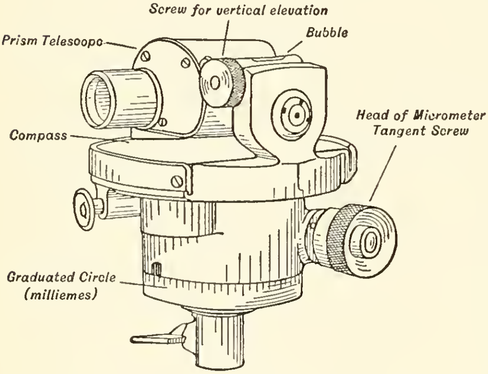

Screw for vertical elevation
Prism Telescope
Bubble
Head of Micrometer Tangent Screw
Compass
Graduated Circle (milliemes)

Fig. 40

Possible Directions of Advance in Design.—It is doubtful whether finality will ever be reached in instrument design. For small theodolites, where angles are required to the order of one or two minutes only, some features might usefully be copied from the small director for field artillery shown in fig. 40. In this instrument angular

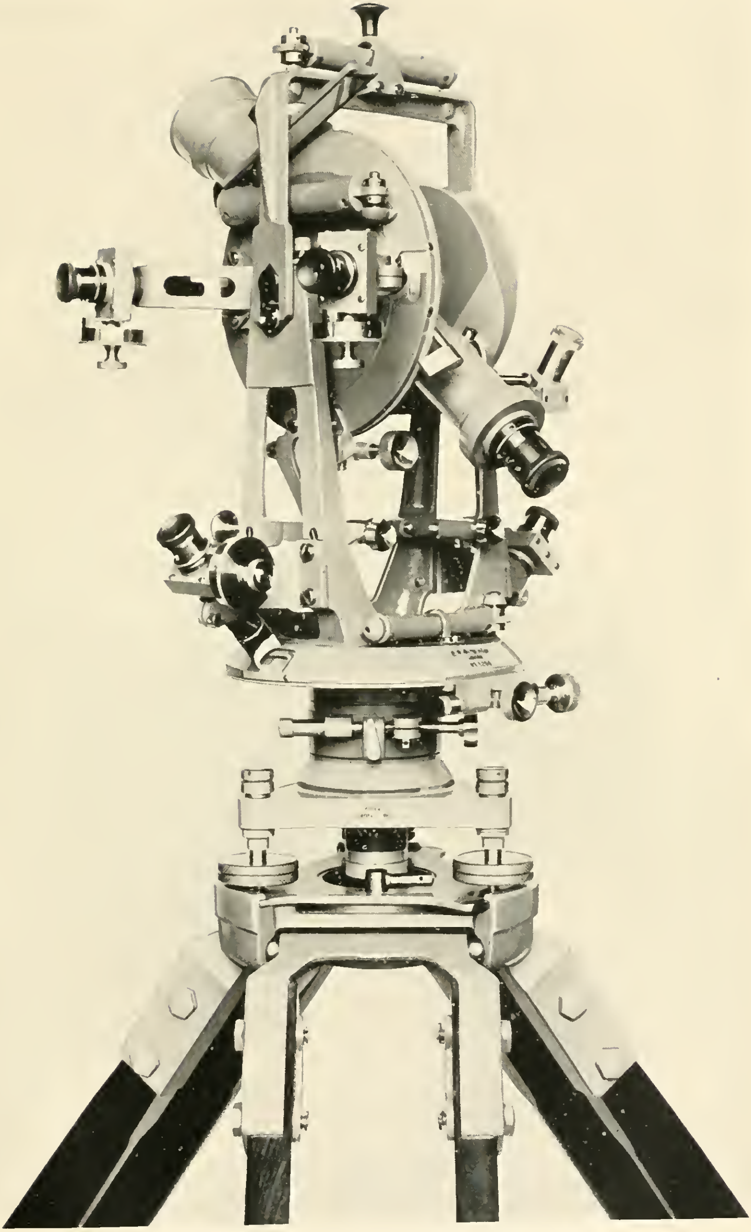


Fig. 41.—Watts' Micrometer Theodolite

rotations are read to 1 millième by a micrometer screw. The worm gearing may be disengaged for large movements by pressing a lever. The dimensions of the telescope (magnification = × 2) are kept down by the use of prisms for erection of the image. Horizontal angles are easily read by this instrument to within 2′ of arc. A vertical scale is provided in the eyepiece for the measurement of small vertical angles, and the object glass is mounted in eccentric rings so that the telescope may be collimated to make the axis horizontal when the bubble on the small spirit level is at the centre of the tube. A compass needle is mounted in the lower portion of the instrument.

For many purposes this simple instrument, once adjusted, will be as suitable as a small theodolite, besides being far more easy to use, as verniers are dispensed with.

Some of the above features have recently been embodied in small theodolites for following balloons in investigation of air currents. Both altitude and azimuth movements are given by worm gearing, and readings can be taken very quickly.

On the other hand, in cases where accuracy is required, the modern tendency is again towards the abandonment of the vernier and the substitution of the micrometer microscope, even with instruments with comparatively small circles. Some tendencies to change in mechanical design are also to be noted, among the chief of which may be mentioned the reversion to the cylindrical vertical axis. Modern methods of precision grinding and lapping make it possible to secure a very exact fit between cylindrical bearings. This device thus destroys all degrees of freedom of the central pin except the vertical translation. This is controlled by a single ball bearing or by a ball race if much weight has to be supported. On the other hand, the *satisfactory* construction of such bearings is even more difficult than those of the conical type; it is difficult to strike the happy medium between the tightness and friction which arises when the fit is too close and the looseness and shaking which ensue with the other extreme. When circles have to be read to a few seconds the allowable shake is extremely small. On the whole it is doubtful whether cylindrical axes are yet to be preferred to those of the conical type when the question is considered from all standpoints.

A Five-inch Micrometer Theodolite.—To conclude the discussion of the theodolite, it may be of interest to describe one of the most modern transit theodolites, an example made by Messrs. E. R. Watts & Son, Ltd. It is illustrated in fig. 41. The levelling screws have their milled heads below the tribrach and the threads are protected by dust caps. The whole upper portion of the instrument can be loosened from the tribrach and given small traversing motions (sliding on a planed annular ring) for centring. The horizontal circle is completely enclosed; the case forms the alidade, carrying the micrometer microscopes and the telescope standards; conical axes are used in the present example. Outside the telescope standards are the windows through which the micrometer microscopes (clamped to the standards) read the circle; these windows are of thin plate glass. The micrometers read by estimation to 1″ of arc. The objectives of the micrometers can be adjusted by capstan screws and the whole microscope can also be adjusted.

The clamp for the alidade is found in the plane of transit. This has several advantages, being well out of the way of the micrometers, but it restricts slightly the angular range of the telescope in positions near the vertical. The shape of the telescope standards will be sufficiently explained by the illustration; as was previously indicated, the standards and cover plate form one casting.

The telescope pivots are of hardened bell metal and are non-interchangeable; they rest in segmental bearings, one side being adjustable by a capstan screw. Flanges on each side of the bearings ensure that any thrust on one standard is communicated to the other.

The vertical circle is again completely enclosed; the micrometer microscopes are clamped to the case, which forms the alidade and bears the spirit level. The alidade moves on the spindle formed by a conical part of the telescope pivot. Since the circle and case form a heavy weight, counterbalancing weights have to be added near the other bearing in the telescope standard.

The telescope has internal focusing and is anallatic; it usually carries stadia lines on a glass diaphragm. The focal length of the objective is 10 in., and the magnifications given by the two eyepieces are 25 and 35 respectively. The telescope carries open sights below it. The bubbles on the instrument give 1 division shift per 10″.

The cylindrical axes of the telescope are, as usual, prolonged slightly beyond the standards to receive the striding level as shown in the figure. A slightly eccentric disc just below the axis serves as a means of adjustment for the striding level.

Transit Instruments.—The *transit circle* or *meridian circle* is an astronomical instrument used for the measurement of stellar positions. It consists simply of a telescope moving on a horizontal axis pointing exactly east and west, so that the optical axis of the telescope swings in the meridian. The general arrangements are similar to those of the theodolite, although there is usually no possibility of rotation round a vertical axis in instruments intended specifically for " transit " observations, the supports of the telescope being generally made exceedingly strong and rigid. For the larger instruments carrying object glasses of 6 in. diameter or more the supports may be placed on piers of masonry, but for smaller instruments pillars of iron are employed.

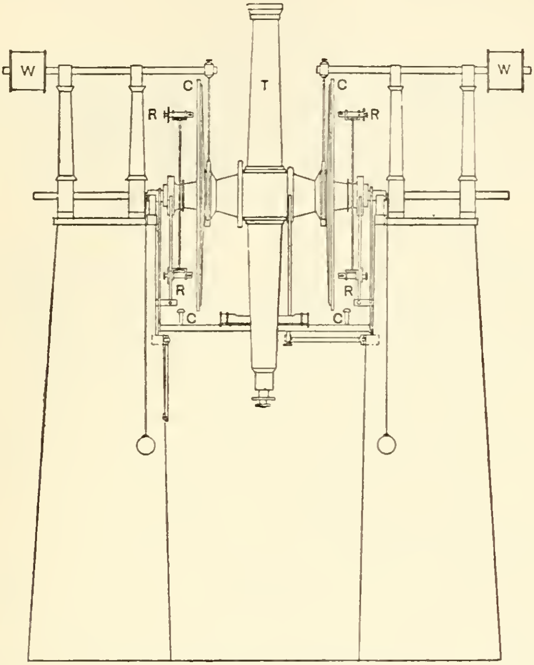


Fig. 42.—A Transit Instrument

T, Telescope. R, Reading microscopes. C, Divided circle. W, Counterbalancing weights. (From Chauvenet's *Spherical and Practical Astronomy*.)

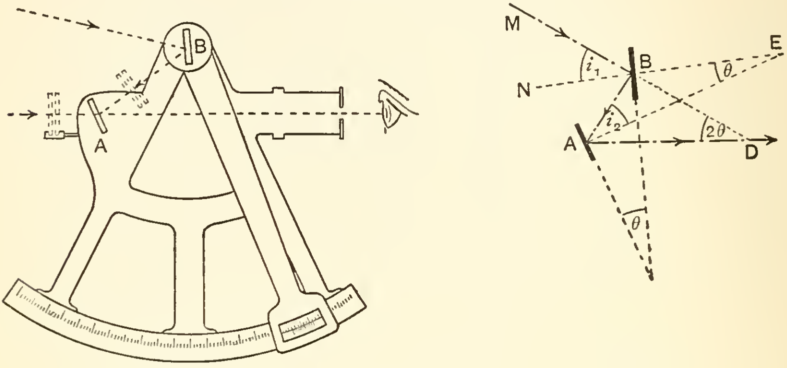
More especially in the larger instruments elaborate precautions are necessary in order to avoid variation in the form of the tube, owing to the varying bending moment exerted by gravity on the tube when this is placed in varying positions. To minimize the effects of this nature the tube of the telescope is often made broad in the central portions (fig. 42).

This construction of the tube is followed with advantage in some of the larger theodolites, where the aim has to be to secure maximum rigidity with the minimum weight.

The errors and adjustments of the transit instrument are similar to those encountered with the theodolite. The adjustment of the horizontal axis of rotation and the collimation of the central vertical cross wire may be carried out by means of an " autocollimating " method, for a description of which textbooks on astronomy should be consulted.

The Sextant

General Principles.—The sextant is an instrument of which the design is the outcome of the necessity for determining the altitude of the sun or stars on board ship. Originally designed by Newton, the instrument was reinvented by Hadley. It takes the place of the theodolite in land observation. Owing, of course, to the motion of a ship it is necessary to have the images of two points (of which it is required to measure the angular substance) in the field of view of the instrument *at the same time* in order that the appreciation of the proper setting may be instantaneous, and therefore not upset by the variations in the position of the instrument.



(a) (b)

Fig. 43

The instrument (fig. 43 (*a*)) consists in its essential form of two mirrors A and B (imagined perpendicular to the diagram) with an aperture (in the plane of the diagram) which is fixed with regard to the mirror A. B can be rotated about an axis (perpendicular to the diagram) in the plane of the reflecting surface. It can be imagined that the mirror B extends a short distance above and below the plane of the paper, while A only extends below it. The eye therefore sees any distant object over the edge of A, and sees another part of the view (by light reflected both from B and A) apparently *through* the mirror A.

When B is accurately parallel to A, *distant* objects seen in the mirror will appear continuous above the edge of A, but as B is rotated the points of the " field " seen in apparent juxtaposition " past " and " through " the mirror will be those which in reality subtend an angle which is double the amount of rotation of B. This is easily proved* from the law of reflection. The subtended angle is therefore conveniently measured by mounting on the alidade the vernier or verniers of a divided circle. The divided circle must, of course, be accurately parallel to the plane of the diagram, and the readings must necessarily be corrected as if the reading were zero when B is parallel to A.

Details of Construction.—Some modern instruments are actually being made with non-corrodable (" rustless " steel) metallic mirrors, but the impossibility of this convenience in times past necessitated the construction of the instrument with mirrors coated on glass with surfaces as perfect as possible. Thus H, fig. 44, called the " horizon glass " from the usual observation through it of the horizon at sea, is silvered over half its surface in order to have a sharp yet protected edge to the mirror, the whole glass being mounted in a square metal frame.

I is also a mirror silvered on the back and mounted in a metal frame screwed down to the centre table of the alidade; it is known as the " index glass ".

Instead of a simple aperture a telescope may be employed, and this must have its axis parallel to the plane of the divided circle. In this case we have two images in the field of view, one due to the direct light through the plain glass of H, and one viewed by light reflected from the mirror " backings " which cover the index glass and half the surface of the horizon glass. These images spread, of course, one over the other in the field of the telescope, but they are easily distinguishable owing to the possibility of moving one of them by rotating the mirror I.

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* See fig. 43 (*b*). Draw normals to A and B intersecting at an angle θ, the angle between the mirrors. Let i_1 and i_2 be the angles of incidence of the rays on B and A respectively. The remainder of the lettering and construction will be understood from the figure. Applying the theorem regarding the exterior angle of a triangle we have:

1. From triangle ABE: $\angle$ NBA $= i_1 = i_2 + \theta$.
2. From triangle ABD: $\angle$ MBA $= 2i_1 = 2i_2 + \angle$ BDA.

Therefore $\angle$ BDA $= 2\theta$.

The telescope is used when it is required to measure the angle between two stars. Without the telescope the images could not be made to appear simultaneously, except just at the edge of the mirror, and it would not be possible to hold the instrument sufficiently steady in the hand or on board ship to make a satisfactory setting of this nature.

Naturally also the use of the telescope will enhance the accuracy of observation, as a " coincidence " will be detected with greater ease; but the magnifying power is not carried high in practice, especially when using the instrument in the hand, as the image is difficult to keep sufficiently still for observation, and moreover this difficulty increases with the magnifying power. In some instruments the alternatives of (1) a simple aperture, (2) a Galilean telescope of power 2, (3) an inverting telescope of power 6, or similar systems are provided.

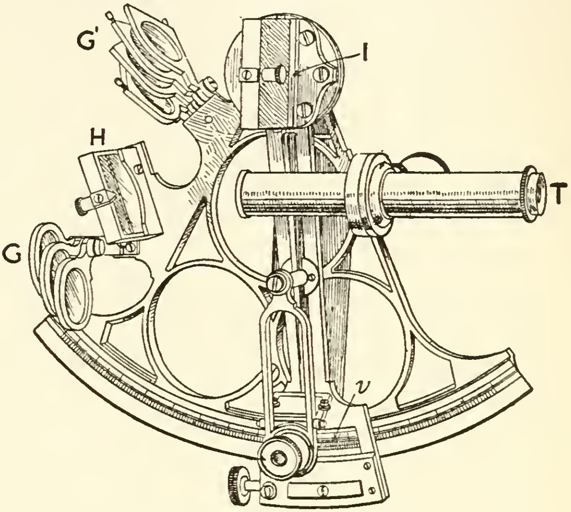

Fig. 44

A very important point is to secure that in any case no light which comes from outside the limits of the horizon glass reaches the eye. Restriction of the size of the telescope object glasses, and the provision of a suitable tube and stop for use with the simple aperture is necessary.

When the horizon glass has been covered with a piece of black paper which does not extend over the metal frame, no bright light should be seen on examination of the exit pupil* with a simple magnifier when the telescope is directed towards a bright source of light.

There is usually an adjusting screw which moves the telescope T at right angles to its own axis in order that the aperture of the object glass may receive a relatively varying amount of light from the two portions of the horizon glass. This is intended for use in equalizing the brightness of two images which it is desired to set in coincidence, but there are good reasons against the too drastic use of the device; when a small part only of the object glass is used for one image the definition is injured, and there may be a certain amount of displacement of the image (owing to any irregular aberration of the lens) which occurs in a manner very similar to that described in the chapter on the spectrometer.

When settings have to be made on the disc of the sun, light filters are necessary, and batteries (G and G′) of three or four filters are provided which may be placed in the paths of the light rays before they reach the index or horizon glass.

General Construction.—The frame of the instrument is generally a bronze casting made to combine rigidity with lightness, and the radius of the circle is about 7 in. to 6 in. in ordinary instruments. The graduations are frequently upon silver, and the vernier (*v*) " reads " to 10″ of arc in good ordinary instruments. The axis is generally very short and dumpy.

As is implied by the name " sextant ", the arc of the instrument covers in reality 60° or so, but the divisions are, of course, only half their nominal amount. In some instruments the arc may be rather greater and numbered up to 160° or 170°.

As only one vernier is provided, it is not possible by inspection of opposite vernier readings quickly to test for eccentricity, a fault to which the sextant would seem, *a priori*, somewhat liable. It is therefore necessary to calibrate or test an instrument for accurate work, and this may be done with the help of a good theodolite by observing from a special point the angular readings of a series of *distant* objects in the field, and afterwards obtaining the relative angles as indicated by the sextant. For the regular testing of a series of instruments, a number of collimators with suitable cross wires may be set up on rigid mounts around a circle of a few feet radius to give a series of imitation distant objects of which the relative angles remain constant. In this way a curve of calibration may be obtained. Such a test is carried out at the National Physical Laboratory.

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* The small bright disc of light always seen on looking at the eyepiece of a telescopic system from say 18 in. behind the instrument when this is directed to the window or the sky.

The presence of eccentricity may be detected if, when the circle is mounted on the dividing engine, a fine circular line is drawn, close to the graduation, having the dividing centre as its *true* centre. A fiducial mark should then be made on the side of the vernier mounting, just to move over the circular line. Eccentricity will then be indicated by relative motion of the mark with respect to the circle, and the amount may be approximately assessed. This is the suggestion of Instructor-Commander Baker, R.N., and has been adopted in some classes of instruments. The workshop practice in testing of centring is, however, simply to ascertain the variation, if any, of the length of the vernier in terms of scale divisions. If the vernier is apparently too short in any position, it has evidently been displaced outwards from the dividing centre, or vice versa. Therefore the centre of rotation of the index arm is displaced towards the point where the apparent length of the vernier in scale divisions is a minimum, or away from the maximum.

The eccentricity error must not exceed 40″ if the instrument is to obtain an " A " certificate from the National Physical Laboratory.

A short geometrical consideration easily shows that the maximum centring errors in the sextant readings will be found when the displacement of the centre is on the line of the centre scale reading. The effects will be a minimum when the displacement is at right angles to this line. The maximum tolerance for actual displacements of the centre in these two directions may amount to about one-thousandth of an inch and three-thousandths respectively for a 7-in. 120° sextant.

Where greater weight is not objectionable it is an advantage to complete the circle and add another vernier, thus forming the " repeating circle ", but in the general design of instruments the requirements of the user must be borne in mind. It is useless to indulge in great complexity and accuracy, with the inevitable enhanced cost of an instrument, when the very conditions of ordinary use would render superfluous the refinements offered.

Where angular measurements of the order of accuracy of one or two minutes only are required, it is possible to gain considerable simplification by abandoning, as has been done in certain military instruments such as dial sights and directors, the use of the vernier, and substituting instead an accurately made tangent screw or worm gearing, so that fractional parts of divisions may be directly read off from a micrometer drum with the naked eye, thereby saving the troublesome estimation of the coincidence of divisions. For this observation with the vernier it is necessary to look through a small magnifier, incidentally obstructing the light with the head and securing the most unsatisfactory relative lighting of scale and vernier, for the vernier of sextants invariably moves *over* the circle. Attention is again drawn to the comments made on this matter in Chapter I.

Instrumental Errors.—The first order instrumental errors to which the sextant is liable may be given as follows:

1. The zero error (usually taken before each reading).
2. Centring error (must be reduced below a certain value in the construction of the instrument).
3. Non-parallelism of index mirror.
4. Non-parallelism of horizon glass. (This causes a constant error in all readings.)

The second order errors are principally due to the three following causes:

1. Want of perpendicularity of the index mirror to the plane of the instrument.
2. Want of perpendicularity of the horizon glass to the plane of the instrument.
3. Want of parallelism of the telescope, or line of sight, to the plane of the instrument.

Adjustments.—The necessary adjustments of the sextant may be made as follows:

1. *The reflecting surface of the index reflector must be perpendicular to the plane of the sextant.* The accuracy of the adjustment may be tested by setting the vernier near the middle of the arc, and observing, with the eye quite close to the reflector, the arc *and* its image seen by reflection in the mirror B (fig. 43).

If the glass leans forward the reflected image will appear high, and vice versa. There is often no easy means of making the adjustment unless the mounting of the mirror is loosened and a small piece of paper slipped underneath in such a position as to tend to correct the error.

2. *The plane of the horizon reflector must also be perpendicular to the plane of the sextant.* When this adjustment is correctly made, the index reflector can be brought accurately parallel to the horizon reflector by moving the index arm, and in this case the direct and reflected images of a star will coalesce.

If the reflectors are *not* parallel the one image will move *past* the other when the index arm is slowly rotated. A small capstan screw is usually provided for altering the tilt of the horizon glass, until the images can be made to coalesce.

The adjustment may be made in the laboratory if a 3-in. aperture collimator is available with an artificial star, or fine illuminated pinhole, at the focus of the lens. The adjustment may also obviously be performed if a fine straight line, such as the sea horizon in very clear weather, can be brought into the field of view. In the latter case coincidence is first secured with the plane of the instrument vertical, and the test of adjustment is then made by turning its plane nearly horizontal, when the images should not separate. The star will, however, be a much better object for making the adjustment.

3. *The sight line of the telescope must be parallel to the plane of the instrument.* This adjustment is easily tested by placing two small metallic sights of equal height on the arc when the instrument is lying on a table. The tops of the sights should be at the same distance from the plane of the instrument as is the axis of the telescope.

It is now possible to make on a wall at some distance, 20 ft. say, a mark which is in line with the edges of the sights and which may be viewed by the telescope. A horizontal line drawn on the wall through the mark should then pass through the centre of the field of the telescope.

There are generally two small screws on the ring which carries the telescope, by means of which its inclination may be adjusted.

This point may be tested in the field by making a coincidence adjustment in the side of the field of view nearer the instrument, and observing whether the coincidence is still perfect when the images have been brought to the opposite side of the field.

4. *The index correction or zero error must be ascertained.* The vernier reading should be zero when the mirrors are set accurately parallel by means of coalescing the images of a star, or artificial star from a very carefully adjusted collimator. In any case the reading is taken, which gives at once the zero error. At sea the setting may be made on the horizon instead of on the star or on the disc of the

sun; in the latter case, however, the dark shades must be used. The introduction of any light filter into the path of the light may cause a modification of this correction, and care must be exercised in this respect.

5. *The effect of the coloured glasses must be known.* If the light filters are at all prismatic a sensible error may be caused in the readings. The glasses may be directly examined separately, when dismounted from the instrument, with a good theodolite or spectrometer and a suitable bright object. If it is necessary to work with the sextant alone the sun's diameter should be measured with a certain combination of shades. One of the glasses may then be reversed and the measurement repeated. Half the difference of the two measures is the error produced by that glass. In sextants made by the firm of Pistor and Martin the shades may be reversed quickly without difficulty, so that automatic compensation may be made by reversing the shades half-way through a set of observations. On the other hand, if the error due to each glass has once been measured it is a simple matter to allow for it in calculation, and probably this would be more satisfactory.

6. *The errors due to the form of the glass mirror reflectors must be ascertained.* Certain errors are possible in the older forms of instruments which do not arise when the metallic reflectors are employed. The chief of these is the possible error due to a prismatic form of the " index glass ".

A little consideration will show that as the rays in all settings follow similar paths through the *horizon glass*, a prismatic form in *this* will affect all measured angles equally, including the " zero " setting, and therefore need not be considered.

The defective index glass may be considered as a combination of a prism of small angle and the reflecting surface. Naturally the only significant part of the error is its component in the plane of the instrument. To test the form of the index glass it is sufficient to observe the movable image of a sufficiently bright star in the sextant telescope of highest power.

Part of the light is reflected at the front face of the index glass, and if a prismatic error is present a ghost movable image of lesser brightness will appear at the same time.

Naturally the ghost image will give the truer readings, and if it were possible to make independent sets of angular measurements taking the zero error, &c., as usual, first with the ghost image, and second with the main image, an estimation of the necessary correction for this error might be made. If, however, the instrument has previously been calibrated so as to detect errors of eccentricity, the effects of the index glass will have been included in the calibration curve if one is provided.

Future Improvements.—Baker* discusses possible improvements in the construction of sextants, and mentions the substitution of worm gearing for the vernier. The spacing of the teeth in some instruments already constructed is such that 720 teeth would be found in the complete circle. The head of the worm is divided into 60 parts which would then correspond to ·5′ each. A quick release lever would have to be provided to disengage worm and teeth for large movements of the alidade. Baker also comments upon the advantages which can be gained by substituting a "right-angled" prism for the index mirror, fig. 45. This prism would necessarily be silvered on its hypotenuse face, which would act as the reflector. Among the chief advantages secured is the maintenance of a finite aperture of the beam up to an angle of incidence on the mirror of 90°, when the breadth of the reflected beam would be zero in the ordinary case.

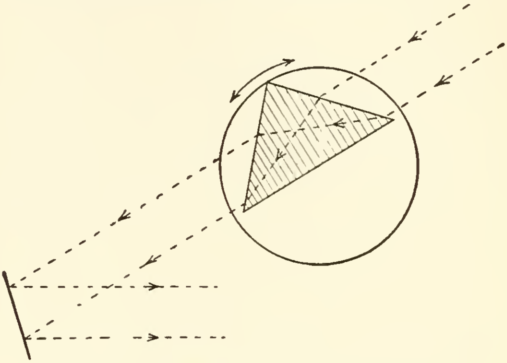

Fig. 45.—Use of Right-angled Prism as an Index Glass

A Special Problem in Angular Measurement: The Position-finder.—In the course of engineering work optical problems of a special nature frequently arise, and these can often be solved by the use of special optical devices. As an illustration may be taken the problem of finding the angle between two straight lines intersecting in the field of view and lying in a plane perpendicular to the direction of view, such as marked by the lines joining one star to two others. One obvious plan is to employ a telescope capable of rotating about its own optical axis. A straight cross wire in the focal plane is made parallel to the two lines alternately, and the rotation is found by a divided circle. Alternatively, as in some types of position-finder, the cross wire* alone may rotate. The *angle*, however, is not magnified by the telescope. Suppose, however, that a simple inverting prism (the action of which is similar to that shown in fig. 45) is placed in front of the object glass so as to cover half the aperture. An erect image and an image inverted with reference to the plane of the base of the prism are found superposed in the field. If we turn the prism *about the direction of view* the inverted image rotates at twice the angular speed of the prism. It would thus be possible to measure the above angle more accurately by turning the prism into the two successive positions required to produce accurately inverted and superposed images, first of one line or pair of stars and then of the other. The rotation of the prism would be measured and there would be no need for a cross wire in the telescope. There are obvious possibilities about this device, suggested recently by Professor Cheshire, in connection with the measurement of departures from perpendicularity of walls, the relative angular positions of stars, &c. Inasmuch as the superposition of like objects conduces to an exact setting, the arrangement appears to be a great improvement on the ordinary turning cross wire.

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* *Dictionary of Applied Physics,* article on "Navigation".

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* A type of position-finder eyepiece in which cross wire and eyepiece both rotate, is shown in fig. 94 in use on an optical bench.

CHAPTER VI

Levels

General Considerations.—One of the most important operations in surveying is known as levelling, and is carried out by special instruments known as levels. These are usually optical sights which by the aid of sensitive spirit-levels can be used to fix the horizontal direction or the direction normal to the gravitational force. By their aid the relative heights, above some standard level, of parts of a country can be found by methods of " stepping " with the familiar surveyor's staff. In its modern form this consists of a staff of hard wood of rectangular section and 10 ft. or more in length, on the surface of which the graduations are stencilled or cut by a special machine similar in principle to the linear dividing engine. The depressions thus cut out may be filled with black, to stand in contrast against the untouched wood, which is painted white. For work of the greatest accuracy the graduation may be made upon a strip of " invar " let into the surface of the staff. In standard surveying work the probable error should not exceed in millimetres the figure obtained by taking the square root of the number of kilometres over which the levelling is performed. Most modern levels are fitted with stadia lines so that in addition to their main intention they may be employed as tacheometers.

One or two simple considerations will elucidate the basic principles of the use of bubble levels. Imagine that a closed glass vessel of no particular shape (fig. 46 (*a*)) contains some liquid, of zero surface tension, which nearly fills it. The surface of the liquid would thus be a truly horizontal plane. It would be possible to mark the line of contact ll' between liquid and glass so that the vessel could always be brought into the same position with respect to the horizontal. Suppose now that the vessel is supported by two legs A and B of unequal length (which are kept in a vertical plane), resting on an arm CD, pivoted at O. If the vessel were turned 180° about a vertical axis and again rested on CD, the liquid would now make a new line of contact with the glass. If the arm CD were rotated about O, the original line of contact could be restored; when this is performed the mean of the two positions of CD evidently marks the horizontal.

Should it be necessary to equalize the legs (or rather the perpendicular distance of their ends from the plane of the contact line) it will be seen that in the present example (fig. 46 (*b*)) we could proceed to correct about half the error by reducing the length of leg B and half by rotation of CD, and so on. When the bubble can be reversed about a vertical axis without changing the contact line the conditions are clearly established that A and B are equal, and CD is horizontal (fig. 46 (*c*)).

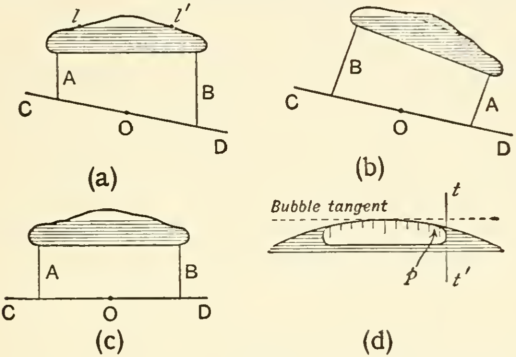

Fig. 46

While these considerations are totally independent of the shape of the vessel, in practice it is necessary for ordinary convenience in working to employ a tube of uniform curvature. A given angular tilt then causes the bubble to move a correspondingly related distance along the tube. Furthermore, since the bubbles are small and the liquids have a definite surface tension, an irregular shape of the upper surface of the vessel would tend to produce, on this account alone, highly irregular movements of the bubble for small differences of tilt, and it would be practically impossible to judge the amount of compensation needed to " correct half the error " as suggested in the case above. It was doubtless this cause which prevented the general recognition of the superiority of " bubble " over " plummet " levels till the process of grinding allowed the production of a sufficiently uniform curvature. It will be realized that if uniform *movement* of the bubble is of such importance the state of the interior surface as regards cleanliness is a matter which requires the utmost care. Nothing which will change the relative surface tension effects between the glass and liquid may be allowed. For this reason the graduations of the bubble have up to the present time always been placed on the exterior of the vessel, thereby introducing the disadvantage of the possible parallax between these divisions and the bubble ends. On account of the surface tension the ends of the bubble are naturally curved, and the " end of the bubble " usually observed is the tangent tt' to the curved surface, which itself is some small distance from the contact point p (fig. 46 (*d*)). As indicated above, the modern optical level consists of a sensitive spirit-level in combination with an optical sight. The main adjustment is to make the line of sight or " collimation " truly horizontal when the bubble is in some definite position in the tube. Once the adjustment has been effected the sighting line may always be made horizontal by a simple setting of the bubble. In referring to the bubble tube it is convenient sometimes to speak of the " bubble tangent ", by which is meant the tangent to the interior of the tube taken at the mid point of the graduations (see fig. 46 (*d*)) and parallel to the axis of the tube. The bubble tangent is horizontal when the bubble is central (in a perfect tube).

It should be remembered that this bubble tangent in the case of a slightly irregular tube would be better specified as: a line between two definite points of the bubble tube which is in the same plane as the axis of the tube, and which becomes truly horizontal when the bubble is brought into the symmetrical position. The bubble is " set " by bringing its ends symmetrically disposed with respect to the centre graduation on the tube, or, as in the Zeiss prism reader, with respect to the separating line of two prisms. Should the length of the bubble vary through changes of temperature, the direction of the bubble tangent is liable to variation unless the figure of the tube is symmetrical with respect to the radius of the slightly curved bubble tube at the mid point. If it is the intention to use the bubble for the measurement of small vertical angles, it is evidently essential in addition that the curvature of the tube shall be uniform throughout.

Types of Instruments.—Until comparatively recently the principal classes of levels were comprised in the " Wye " and " Dumpy " types. In the first the telescope is carried in two Y-shaped supports mounted on the stage, the tube being provided with two equal cylindrical flanged collars at the places of support. The level tube is usually mounted on the telescope, adjusting screws being placed at one end. The stage rotates about the usual conical vertical axis in the tribrach, and a clamp and fine adjustment for the azimuth motion is provided. This type of instrument has passed largely out of favour, and in many cases its manufacture has been discontinued, but to some extent it led the way for the development of the Zeiss level. In adjustment the line of sight had first to be made parallel to the axis of the telescope collars. If the adjustment was incorrect the cross-wire intersection would be displaced in the field of view by a rotation of the telescope through 180° about the collar axis. The bubble tangent was then adjusted parallel to the collar axis, the first test being that if the bubble was in adjustment for a given position of the telescope, the adjustment would not be affected for a slight rotation of the telescope in the collars if the bubble tube axis is in the same plane as the collar axis. The adjustment for true parallelism of these axes was made by reversal of the telescope in the Y supports. Since the line of sight was made parallel to the collar axis it must then be parallel to the tangent of the bubble tube. Finally, the collar axis was made perpendicular to the vertical axis of rotation by a method similar to that which will be described in connection with the dumpy level.

Such were, very briefly described, the steps in adjustment of the Wye instruments. It was found, however, that for constancy of adjustment it was better to have the telescope rigidly connected to the vertical axis. Hence the origin of the dumpy level which is now in wide-spread use. Modifications of the dumpy level have been introduced, the main improvement being the provision of a strong horizontal axis for the telescope, about which it may be given extremely small rotations in altitude by the aid of a micrometer screw, in addition to the usual vertical axis. This change allows of somewhat greater convenience in use, but both the ordinary and improved dumpy types require to be moved from one station to another during adjustment.

One of the greatest advances in instruments for levelling was the introduction of the Zeiss level, which makes use of a rotating telescope (as in the Wye construction) and also embodies other new and valuable features amongst which the method of observing the bubble is noteworthy. It is proposed now to describe an improved dumpy level and the Zeiss level in greater detail.

Engineers' Precise Level.—Fig. 47 represents an improved engineers' precise level (resembling the dumpy type) constructed by

Messrs. Cooke, Troughton, & Simms, Ltd. The construction is indicated in the diagram, fig. 48. In fig. 47 the bubble tube will be seen mounted parallel to the axis of the telescope with the inclined reflector, by means of which the bubble may be observed from near the eye end of the telescope. The latter has an internal focusing system and carries a glass diaphragm with stadia lines ($\frac{1}{100}$). Since the diaphragm is mounted in a special cell for the purpose, it may be removed for cleaning. The telescope body is of duralumin, an alloy of aluminium which combines lightness with a greater rigidity than pure aluminium can afford. The telescope carries the main bubble, and the axis of the bubble can be adjusted relatively to that of the telescope by screws at one end of the bubble tube. A ball-and-socket joint supports the opposite end of the latter.

The tribrach, of the usual type with dust covers for the levelling screws, carries the vertical conical axis of the instrument. A clamp and fine adjustment are provided for slow motion in azimuth. A small circular level mounted on the tribrach serves for the approximate adjustment of the vertical axis. This instrument differs from the simpler levels of the dumpy type in the possession of a horizontal axis about which the telescope may be given *small* rotations in altitude. The rotation is effected by means of a micrometer screw with a drum of the "sleeve" type seen in the figure over the dust cap of the right-hand levelling screw. Ten complete turns of this micrometer drum are equal to a reading of 1 ft. vertical interval on a staff 100 ft. distant; one division of the drum corresponds to about 4 sec. of arc. The motion in altitude is controlled by a spring mounted on the other side of the vertical axis and this spring is sufficiently powerful to prevent back-lash in the screw.

One of the advantages claimed for this instrument is that the bubble in its position at the side of the telescope is less liable to disturbance through unequal temperature of the supports. The sensitiveness of the bubble is about 20 sec. per one-tenth of an inch.

Some levels of this type carry a magnetic compass mounted on the side of the telescope opposite to that carrying the bubble. A reflecting device permits the reading of the compass merely by a slight motion of the eye from the eyepiece end of the telescope.

In levels of all kinds the method of observing or reading the bubble is now recognized to be of very great importance. While in older types no special device was provided, the bubble being simply mounted over or at the side of the telescope, this gives room for (1) disturbance of the instrument when the observer walks round it to observe the bubble, and (2) the possibility of parallax in reading the bubble. There is, in fact, a possibility of parallax even when special forms of mirror are provided for reading the bubble, except in special arrangements as exemplified by the Zeiss prism readers.

Adjustment of "Dumpy" Levels.—Although the addition of the horizontal axis makes the level described above somewhat more easy to adjust than those of the simple dumpy construction, the procedure for the simpler instruments will be described here as it exemplifies the fundamental principles for the work. Three main "axes" or lines must be remembered, the line of collimation * of the telescope, the bubble tangent, and lastly the vertical axis of rotation. The most important adjustment is to make the bubble tangent parallel to the collimation. This can be done without reference to the vertical axis, but it is decidedly convenient to adjust the instrument in the first place so that the bubble tangent is perpendicular to the axis of rotation. If then the instrument is set up so that the latter is vertical, the bubble remains central for any position in azimuth of the telescope.

The level is set up and clamped firmly on its tripod so that the vertical axis is as nearly true as possible. The telescope is brought over two of the foot screws and the bubble is centred by one of these two foot screws only. The telescope is then turned in azimuth through 180°. If the bubble remains central the first part of this adjustment is complete, but, if not, the error is corrected half by one of the same foot screws and half by the capstan screws which control the axis of the bubble tube. The process is repeated until the adjustment is perfect, and the operation has then to be repeated with the telescope at right angles to its previous position, the foot screw under the telescope now being operated. The bubble tangent will, on completion of the adjustments, be perpendicular to the axis of rotation and the latter will be vertical.

A small cross bubble usually provided on dumpy levels may now be adjusted. Also the orientation of the cross wires in the diaphragm may be corrected, if necessary, by reference to the necessarily horizontal lines traced out in the field of view by image points when their displacement is caused by rotation of the instrument round the vertical axis.

Adjustment of the Collimation of the Horizontal Cross Wire.—The essence of this adjustment is this: granting that the vertical axis of rotation can be made truly vertical, then if the instrument were set up exactly in the centre of a room the telescope would point to a point on one wall which is exactly in the same horizontal plane as is the corresponding point on the opposite wall to which the telescope will point after rotation through 180°. A condition to ensure that the telescope shall, in effect, rotate about a truly vertical axis is that the bubble should be exactly in the same position in the tube for each observation. Messrs. E. R. Watts & Sons, Ltd., give the following instructions for the procedure:

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* The effect of the variation of focusing on the line of collimation of the instrument has already been discussed (p. 68).

(*a*) Select a piece of fairly level ground, 300 to 350 ft. in length, and drive in two stakes about 300 ft. apart. Set up the instrument exactly half-way between them so that the two stakes and the centre of the instrument are all in a straight line. Carefully level the instrument, place a levelling rod upon one of the stakes and take an exact reading with the level, then place the rod upon the other stake and drive it (the stake) * in further until the reading is identical with that obtained when the rod was held upon the first stake; we have now two stakes exactly level.

(*b*) Now select a point in a line with these two stakes and from 25 to 50 ft. beyond one of them, and set up the level at this point. After carefully levelling the instrument place the rod upon the nearer stake and carefully take the reading, taking care that the air bubble is in the centre of its run; now place the rod upon the farther stake, and if the reading is exactly the same as before (the air bubble being in the centre of its run) the collimation is correct; if the latter reading disagrees with the former, correct the greater part of the discrepancy by means of the top and bottom diaphragm capstan head screws and repeat the operation (*b*) until agreement is obtained.

The foregoing adjustments suffice to bring the level into adjustment in all respects. Fig. 49 will serve, perhaps, to make the principle more clear. It illustrates the relative positions of level and staves in the first and second parts of the operation. The equal heights of the intersections of the line of sight with the staff in two positions fixes a horizontal line. In the second position of the level the bubble is adjusted after the telescope has been made to sight along a truly horizontal line. An alternative method, again involving two positions for the instrument, this time close up to each position of the staff, is described in textbooks of surveying. The method involves only observations on objects at the same distance and does

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* It is clearly not necessary to drive in one stake farther if allowance is made for the difference of readings when observation (*b*) is taken.

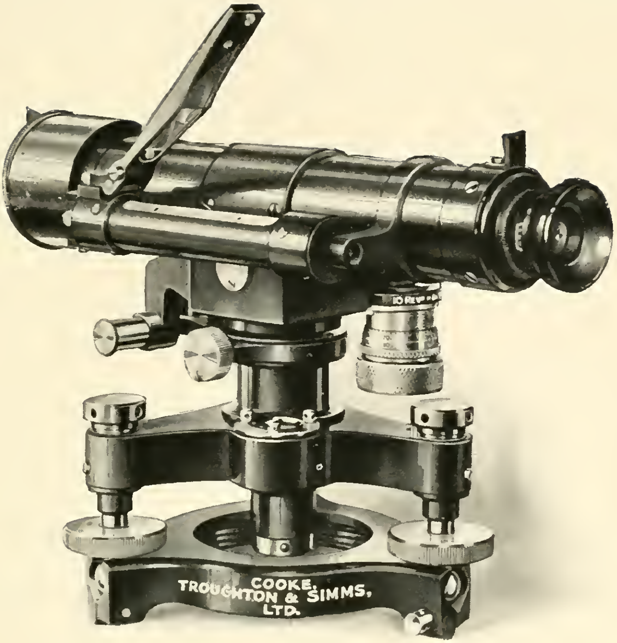

Fig. 47.—Engineers' Precise Level

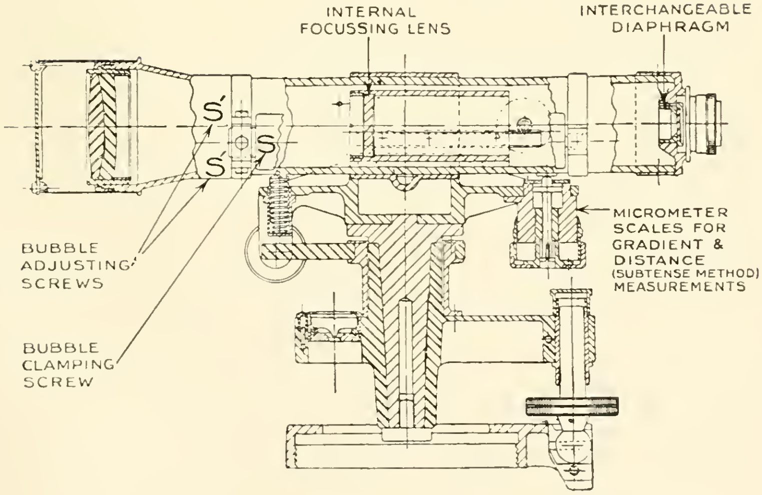

Fig. 48.—Engineers' Precise Level (Diagrammatic)

not therefore involve any change of focus of the telescope, a possible cause of variation in collimation. This method, or a modification of it, is the one recommended by Messrs. Cooke, Troughton, & Simms, Ltd., for the adjustment of their improved engineers' level.

The method of testing for variation of collimation in focusing is to place three stakes in positions A, B, C respectively all in one line. The true difference in level between A and C is now found by the instrument placed midway between them; likewise for B and C. The instrument is then taken to a point beyond all three stakes where some fair range of the focusing movement is necessary to sight upon a staff held on each stake in turn. From this point the adjustment may be made as before in (*b*), so that the true difference of level, if any, between B and C is correctly indicated. If now on adjustment of the focus the level of the nearest stake A is not given correctly, a variation of collimation in focusing may be suspected.

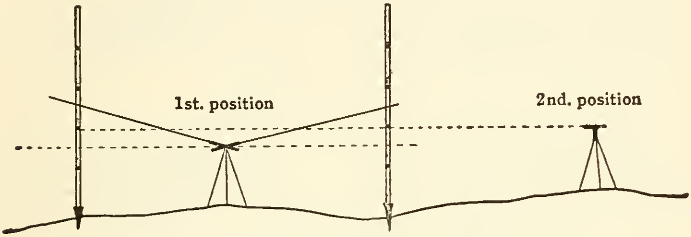

Fig. 49.—Adjustment of Dumpy Level (diagrammatic)

The "Zeiss" Level.*—It will be easily recognized that the factor of time is one of considerable importance in the process of adjustment of a level, both as regards accuracy and convenience. If, for example, the level on its stand settles at all into the ground during the process (*a*) above, the accuracy of the adjustment would suffer because the line thus determined would not be truly horizontal. Furthermore, the necessity of walking about during the operation is by no means desirable. The introduction of the Zeiss level, which can be rapidly and completely adjusted from a single station, therefore marks, for some purposes, a great advance. It was originally due to Heinrich Wild, of Glarus, a well-known Swiss engineer who worked out the instrument in collaboration with Messrs. Carl Zeiss. As in the Cooke level described above, the fine adjustment is effected by a micrometer " gradient " screw which tilts the telescope and bubble through small angles in altitude. Before describing the mechanical construction of the instrument, however, a theoretical system may be considered.

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* Levels of this type are now constructed by British makers.

The instrument employs a telescope which can be rotated about an axis coincident, as nearly as possible, with its own line of collimation. By the aid of a special optical device, described below, the direction of sighting through the tube of the telescope can be reversed. Hence the optical pointer can be used in either direction, but the line of collimation is not necessarily the same in the two cases. A sensitive bubble is mounted on the outside of the telescope. When the latter is rotated through 180° about the axis previously mentioned, the bubble now moves against the opposite side of the glass tube. Hence a new bubble tangent will be concerned. The following treatment is due to Major E. O. Henrici, R.E.

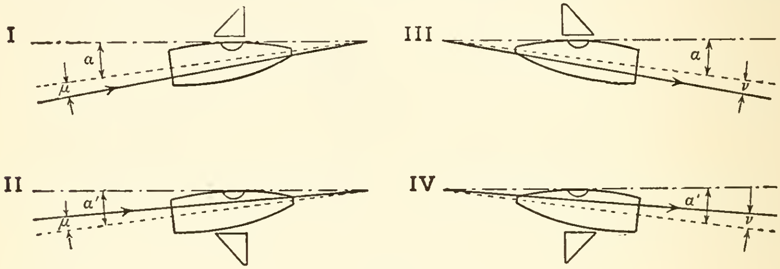

Fig. 50

In fig. 50 the line of alternate dots and dashes represents the bubble tangent, which is always horizontal when the bubble is "set". The line of dashes represents the mechanical axis of rotation, and the thick line the direction of the line of collimation. Let α and μ be the angles between the bubble tangent and the axis of rotation, and between this mechanical axis and the line of collimation respectively. Then in fig. 50, I, the angle with the horizontal made by the line of collimation is A_1, say, where

$$A_1 = \alpha + \mu.$$

The bubble is observed by a prism. In the diagram this serves to indicate the side of the tube which is uppermost.

The telescope is now rotated through 180° about the above axis. This brings a new bubble tangent into consideration, which makes an angle α' with the mechanical axis. The angle with the horizontal now made by the line of collimation is clearly (as shown by fig. 50, II)

$$A_2 \text{ (say) } = \alpha' - \mu.$$

The direction of sighting through the telescope tube is now reversed, and the whole instrument turned 180° on the *vertical* axis. The bubble tube is in the same position as in I. The angle between the new collimation line and the mechanical axis of rotation of the telescope is ν. From fig. 50, III, we see that the angle with the horizontal now made by the collimation is

$$A_3 \text{ (say) } = -\alpha - \nu.$$

On turning the telescope 180° about the mechanical axis as before, the final angle of the line of collimation with the horizontal becomes

$$A_4 \text{ (say) } = -\alpha' + \nu.$$

The mean of the four angles made with the horizontal by the line of collimation is therefore zero. This conclusion is independent of any assumption regarding the regularity of the bubble tube, provided only that the bubble tangents remain constant during the operation. In practice, readings are taken on a vertical staff in each of the four positions of the instrument. It then follows that when the telescope is directed to the position on the staff given by the mean of these readings, the line of collimation becomes truly horizontal.

While the above treatment shows that the bubble tubes need not essentially possess exactly uniform or equal curvature, nevertheless from the practical point of view these qualities are important. In practice two distinct tubes might be used, but the top and bottom of a single vial are made to serve. If the curvature of the interior surface of the tube is irregular the bubble may tend to show a corresponding irregularity in movement. Furthermore, as previously pointed out, changes of temperature during the adjustment of an instrument are likely to cause changes in the length of a bubble unless this is one of a specially compensated type (p. 66).

General Construction of Zeiss Level (Type 2).—The instrument is pictured in fig. 51 and is seen mounted on a substantial stand. The usual three screws with dust covers support the tribrach to which the upper part of the instrument is fitted and clamped. (These parts are packed separately when the instrument is taken down.) The central support of the telescope is seen to be of large diameter, but the outer part is a hollow tube or shell casting rigidly fixed to the telescope. Passing through it is the micrometer screw which bears against the top of a lever pivoted below. The short arm of the lever acts against a bearing surface connected with the sleeve of the central axis. The action of the micrometer screw and lever is opposed by a buffer spring. The vertical axis in this instrument is cylindrical, the inner solid member being enclosed by a perfectly fitting cylindrical sleeve (cleared away in the central parts) through the upper end of which passes the horizontal axis about which the telescope and shell casting are pivoted. The weight of the telescope and sleeve is borne by a single ball bearing of hardened steel carried in the upper end of the inner cylinder. The screw on the left passing through both sleeves prevents the outer parts from being lifted off the axis. The outer shell casting acts as a crank to the telescope, which can thus be given very small

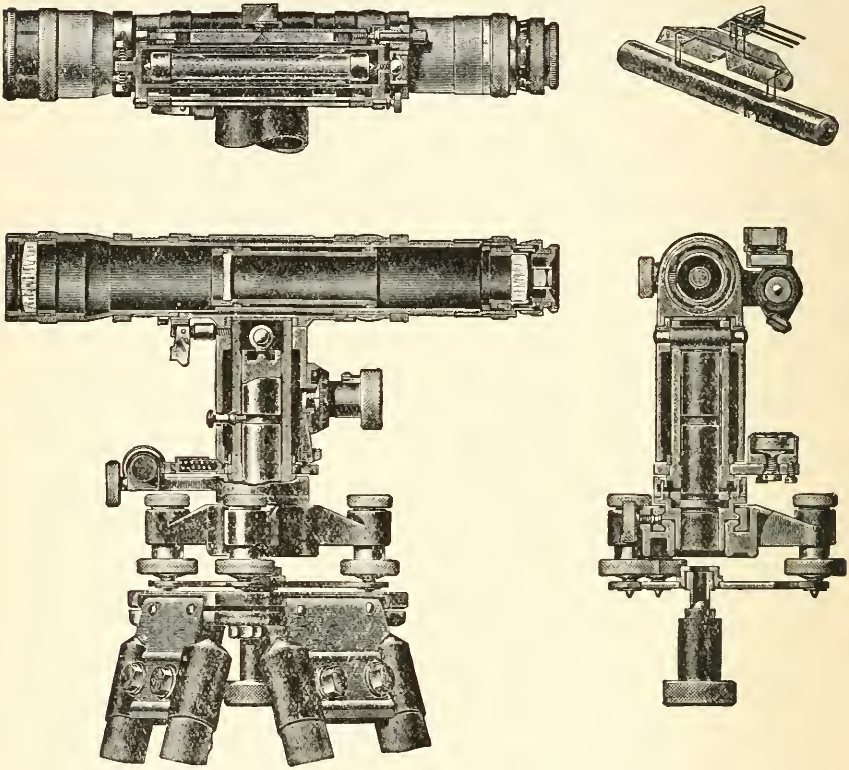

Fig. 51.—The Zeiss Level (Type 2)

adjustments in altitude by means of the elevation and depression screw, sufficient clearance being allowed within the casting for this purpose.

Attached to the sleeve of the axis is a small circular bubble by means of which the vertical axis may be given an approximate adjustment. On the side of the outer casting opposite to the elevation screw is seen the clamp and fine adjustment for small movements in azimuth.

The bubble tube is seen held parallel to the axis of the telescope by a pair of brackets, the left-hand one of which is provided with capstan screws serving to give very small adjustments to the axis of the bubble. The telescope is mounted in ring bearings so that the whole bubble tube and telescope can be rotated through 180° about the telescope axis. When this is done the bubble of air naturally moves against the side of the bubble tube which was first lowermost. The prism system for "reading" or rather "setting" the bubble is contained in the long rectangular casing seen in the figure just above and parallel to the bubble tube. The projection in the centre is a small rotatable casing also containing a prism into which the observer can look from near either end of the telescope. The bubble-observing device, which is one of the most important features of the Zeiss level, is illustrated diagrammatically in fig. 52. It will be remembered that in the theoretical arrangement discussed above it was required that the bubble should always be "set" to a definite location, and the prism observing system accomplishes this in a manner which will be evident from inspection of the ray tracks. If we imagine the compass directions to be as shown in fig. 52 (*a*) the ray from the point B passes vertically upwards, then east after reflection at 1, then north after reflection at 2, vertically upwards after reflection at 3, and finally south after reflection at 4. The image of the length of the bubble thus lies in the figure north and south after reflection at 1, then east and west after reflection at 2

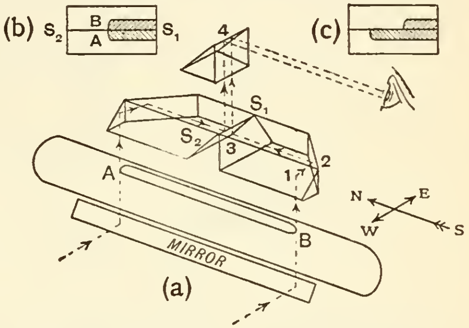

Fig. 52.—The Bubble Observation System of the Zeiss Level

and the succeeding faces. Hence its image is parallel to the dividing line S_1S_2 of the lower pair of prisms. The appearance in the upper prism is therefore as indicated in fig. 52 (*b*); one half of each end of the bubble is seen, and the two ends A and B of the bubble in the tube are evidently symmetrically placed with respect to the dividing line S_1S_2 of the prism system. Should the bubble move, the appearance is shown in fig. 52 (*c*). Further, the two halves of the bubble could be brought together by moving the whole prism system parallel to the bubble tube, should this be at any time necessary. An adjustment for this motion is, in fact, included in the mounting of the prisms in the Zeiss instrument. The rays by which the ends of the bubble are seen cannot possibly make large angles with each other, and the images of the bubble formed by the prism system are equidistant from the eye. Hence the possibility of parallax is entirely eliminated, while the setting of the bubble ends (reminding one of the setting in a range finder) is one of the best possible type. The mirror underneath the tube supplies ample light for observation. Naturally when the bubble system is reversed the air bubble is on the opposite side of the tube while the prisms are found below. This, however, makes no difference to the ease of observation.

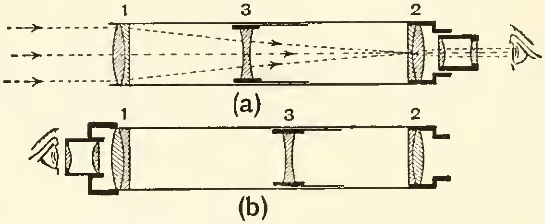

Fig. 53.—Reversible Telescope of the Zeiss Level

The bubble is viewed by transmitted light, and the walls of the tube are blackened at the sides. This results in the greater contrast in the image of the bubble.

The telescope must now be considered. In our theoretical system one of the conditions was the reversibility or approximate reversibility of the direction of sighting, and the manner in which this condition is secured is shown in fig. 53. The telescope is seen to carry an object glass at each end, and in fig. 53 (*a*), object glass No. 1 is forming an image, with the aid of the concave focusing lens 3, in object glass No. 2. The latter lens being placed in the image plane thus exerts little effect on the optical action of the system.

A cap is provided for object glass No. 1, and this cap, by the removal of a central portion, forms an adaptor by means of which the eyepiece, removed from its first position, may be held at the reverse end of the telescope as shown in fig. 53 (*b*). The functions of object glasses Nos. 1 and 2 are now simply interchanged.

The cross lines necessary in the focal plane at each end may thus be placed upon the interior cemented surfaces of 1 and 2. It is naturally the endeavour to make the axes of collimation, so formed, as nearly coincident as possible. The influence of the negative internal focusing lens on the collimation of an optical sight has already been discussed (p. 68).

Use and Adjustment of the Zeiss Level.—In order to use the instrument it is necessary simply to set it up on its stand and bring the vertical axis approximately into adjustment by use of the foot screws and the small circular bubble. Having focused the cross wires with the eyepiece, the instrument is directed to and focused on the staff. The main bubble may then be adjusted by the elevation screw until the " half-ends " of the bubble seen in the prism are in coincidence, and the reading on the staff is taken.

The adjustment of the collimation can be done in four or five minutes without a change of position by the observer. Four staff readings are taken, *each one immediately after a careful adjustment of the bubble coincidence by the elevation screw.*

1. With the telescope in its normal working position. Bubble to left of observer.

2. With the telescope and level turned over through 180° about the telescope axis. Bubble to right.

3. With the eyepiece reversed and the telescope turned 180° in azimuth. Bubble now to left.

4. With the eyepiece as in 3, but the telescope turned back through 180° about its axis into the normal working position. Bubble to right of observer.

The mean of four such readings is free from instrumental errors. The telescope is turned back into its normal working position and the eyepiece is replaced. The telescope is then directed to the position on the staff representing the mean reading as found above. This usually destroys the alignment of the ends of the bubble, but the coincidence can be restored by loosening the locking nut of the prism system over the bubble and giving the requisite displacement of the dividing line by the adjusting screw as suggested above (p. 98). If the motion required is too large, the capstan screws for the coarse adjustment of the axis of the bubble tube by the makers may be called into service. Once the bubble has been adjusted the instrument is ready for use.

It is recommended that for very exacting work the mean reading in positions 1 and 2 above shall be found (even after adjustment) and taken as the true level. If this practice is to be adopted the *adjustment* of the prism reader must be such that the mean reading in positions 1 and 2 (if these are perceptibly different) agrees with the mean reading in positions 1, 2, 3, and 4.

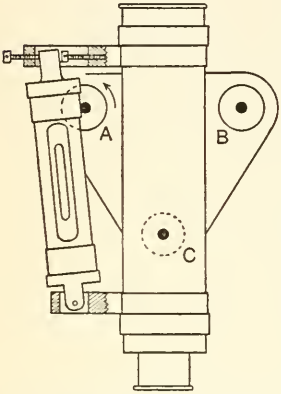

Fig. 54.—Plan of Zeiss Level (diagrammatic)

Some Details in the Adjustment of Reversible Levels.—For the perfect adjustment of the instrument the axis of symmetry of the bubble tube must be truly parallel to the line of collimation of the telescope *as seen in plan*. This adjustment is usually carried out by the makers, but it may be briefly described here. By reference to fig. 54 a capstan screw will be seen at the left extremity of the bubble-tube mount by which it can be given a very small motion in azimuth. A rough adjustment can be given (*after* the above adjustments) by rotating the system through 90° about the axis of the telescope, thus bringing the bubble over the top of the latter. The bubble may then be brought into the " coincident " position by means of the above capstan screw. This, however, would merely make the bubble axis roughly parallel to the axis of rotation, which is not exactly the result required. It must therefore be asked, how the system can be given a small rotation truly about the axis of collimation. Imagine the tripod (fig. 51) to be set up with one leg towards the staff and the instrument directed to the staff and levelled as usual. It will be seen that the three levelling screws of the tribrach are now situated in the positions A, B, C (fig. 54). The exact staff reading is taken. Screw A is then given, say, one whole turn; this lowers one corner of the tribrach and sends the bubble out of centre while also lowering the staff reading. Screw B is then given about one whole turn in the opposite direction so as to reproduce exactly the same staff reading as before. The net result has evidently been the production of a small rotation about the axis of collimation. In this case the adjustment of the bubble will be affected unless the axis of the bubble tube is accurately parallel to the line of collimation. The bubble ends must, if necessary, be brought back into " coincidence " by use of the horizontal capstan screws.

Parallel Plate Micrometer.—In the field of view of the level the image of the staff is seen vertical, and traversed by the horizontal cross wire. The exact reading is then usually taken by estimation to a tenth of one division. In order to prevent inaccuracies of estimation, the larger levels by several makers are provided with an accurately plane parallel glass plate (situated in front of the object glass) which can be rotated about horizontal trunnions by means of a micrometer screw acting on a " bell crank " lever which in its turn causes the displacement of a connecting rod. A hyper-hemispherical knob at the end of the rod engages with a socket on the exterior of the mount of the plate, and thus a motion of the rod causes a rotation of the former. This is the mechanical arrangement in the Zeiss level.

The ray passing along the line of collimation of the instrument is displaced parallel to itself through an interval depending on the rotation of the plate, and the amount of the displacement can be read off on the drum of the micrometer screw, which is directly graduated for this purpose. It will be understood that the motion of the micrometer necessary to bring the cross line exactly to the lower staff division gives the fractional reading required. The drum reads by estimation to $\frac{1}{100}$ mm. or to 0·0001 ft., and this accuracy can be made use of in every exact work.* Some increase in the accuracy of setting is obtained in the Zeiss instrument by providing a split horizontal wire in the left of the field.

Simplified Reversible Levels.—In the discussion of the theory of the Zeiss level it will have been noticed that the reference lines, to which the bubble must be set in each case, were chosen quite arbitrarily. It may easily be seen that positions for these lines could have been specially chosen so that the two bubble tangents are parallel. If this were really the case, the mean of the first two readings would clearly give the horizontal direction. In fact it would be possible simply to adjust the sighting line so that the same point of the field was bisected after rotation in a manner similar to that employed in the simple Wye level, and the line of sight would then be truly horizontal.

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* With the larger Zeiss levels the probable angular error of a level setting may be as small as 0·25 sec. of arc.

Comparatively recently a number of trials have been carried out with instruments employing specially graduated reversible bubbles. The graduation is such that effective rotation of the optical sight about a truly horizontal axis is ensured. The adjustments are similar to those of the Zeiss level, but only the positions (1) and (2) (p. 94) are used. Hence the reversible eyepiece is unnecessary. It is claimed by Connolly * that the quality of bubble tubes and the possibility of accurate graduation by manufacturers is sufficient to justify the production of practical levels which will be sufficiently true for all ordinary purposes. A number of instruments have been produced by various makers for trial of the principle, and it seems *a priori* possible that some uses may be found for such quickly adjustable patterns. It is true, however, that the user of such an instrument will be largely dependent on the maker, and it would be advisable in the first place very carefully to check the indications of a new instrument by treating the whole as a simple dumpy level and carrying out the procedure detailed on a previous page.

Messrs. E. R. Watts & Son, Ltd., have introduced a level of this last type in which one of their non-variable bubbles is employed. It is thus necessary to observe one end of the bubble only for each setting.

Some General Points.—The introduction of the Zeiss level marked (as was said before) a great advance in design, and perhaps of all the novel features therein embodied, the prism reading system and the means which this provides of altering the "symmetrical position" of the bubble by displacement of the prism box, are the most valuable. The adjustment is "coarse" from the mechanical standpoint and therefore does not require any great delicacy of handling (as is the case with many adjustable parts in surveying instruments), yet at the same time a comparatively large movement of the prism box, say 0·5 mm., only corresponds to an angular correction of a few seconds of arc. Furthermore, the arrangement will keep "in adjustment" for long periods. Other features of the "prism reader" have already been specified.

Very frequently, however, especially in "precise" levelling for standard surveys, the methods of observation are such as to eliminate the effects of errors in the adjustment of an instrument (as when equal lengths of sight are taken from a middle station). Thus it may be open to question whether the additional complication of the rotating and reversible telescope is necessary or desirable in every case. There seems to be a considerable field for the improved levels of the dumpy type, and it will also be seen whether the simplified reversible levels prove efficient in practice. H. Wild, the designer of the Zeiss level, has now designed a small simplified level, built somewhat on the lines of the original level, but without a reversible telescope. The original prism reading system, as described above, is, however, retained. The instrument is manufactured by a Swiss firm.

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* *Trans. Opt. Soc.*, Vol. XXV, No. 1.

CHAPTER VII

Range-finders

The General Problem.—The problem in range-finding is to determine the distance of an inaccessible object, an operation which is of frequent occurrence in surveying. In the latter case, at least, the procedure is as follows. By means of a chain, a base (AB, fig. 55) of suitable length is measured out in a direction approximately perpendicular to that of the object O, and the angles OAB and OBA may be measured by a theodolite or other suitable instrument. The triangle may then be solved by the aid of the well-known relation,

$$\frac{\text{OA}}{\sin\text{OBA}} = \frac{\text{AB}}{\sin(\text{OAB} + \text{OBA})}.$$

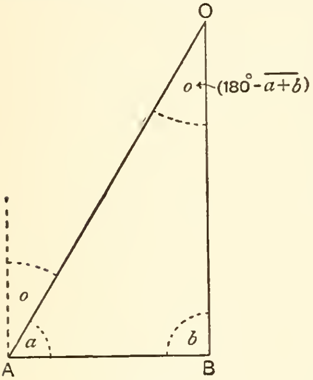


Fig. 55

The problem in range-finding with a single instrument is to find the range OA without the use of a long base. Suppose that (in the original case) AB is equal to 20 m., and that the angles OAB and OBA are measured by the theodolite to 1′ of arc; then if AB has to be reduced to 1 m., the angles must be measured to a far greater degree of accuracy (much beyond the power of the theodolite) to attain the same accuracy in the determination of the range.

Measurements involving this degree of accuracy can be made by the help of the range-finder system, although there is no *direct* determination of the angles. One angle (b) will usually be fixed by the construction of the instrument, say at 90°. The error in range due to an error in measuring or estimating the other angle may be found as follows:

$$\mathrm{OA} = \mathrm{AB} \, . \, \frac{\sin b}{\sin(a+b)},$$

$$\frac{\partial \mathrm{OA}}{\partial a} = \mathrm{AB} \, . \, \frac{-\sin b \cos(a+b)}{\sin^2(a+b)}$$

$$= \frac{\mathrm{OA}^2}{\mathrm{AB}} \, . \, \frac{-\cos(a+b)}{\sin b}.$$

Now in actual range-finding, if b is 90°, a is so nearly 90° that $\cos(a+b)$ may be taken as equal to -1 sufficiently nearly. Hence

$$d\mathrm{OA} = \frac{\mathrm{OA}^2}{\mathrm{AB}} \, da \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Thus the error due to a wrong determination of angle varies as the square of the range, and inversely as the base length. The *percentage* error on the same account will be proportional to the range and therefore does not remain constant.

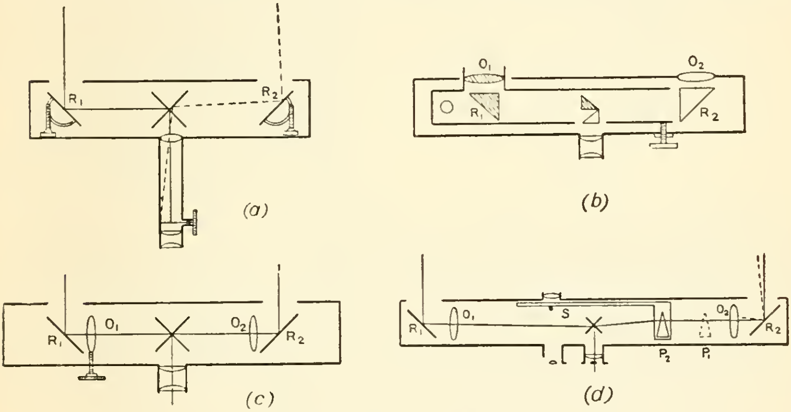


Fig. 56.—Early Types of Range-finder

Range-finder Systems.—In essentials most range-finders consist of two telescopes with object glasses each supplying, by means of reflecting devices, a part of the image which is seen with a single eyepiece. The separation of the view-points is secured by the aid of the end reflectors, which were simple mirrors in early types and are now generally replaced by " pentagonal " reflecting prisms.

Fig. 56 (*d*) represents a Barr and Stroud range-finder of an early type, and contains some of the features found in more modern instruments. R_1 and R_2 are reflectors at 45° to the axis of the instrument, and O_1 and O_2 are the object glasses which focus their images on the crossed reflectors in the centre of the instrument. These reflectors throw the light into the eyepiece, and this arrangement thus produces a field of view divided into two portions by a horizontal line of separation, the images in the upper and lower portions being supplied by different object glasses. In practice the centre reflectors are not simple mirrors but a prism combination. If the rays arriving at the two windows from any point of a distant object were exactly parallel, the two halves of the field of view would exactly correspond; but if this is not the case, owing to the finite distance of the object, there will be a slight relative displacement between these halves.

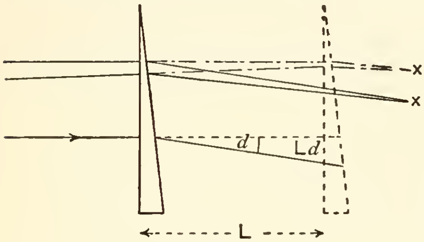


Fig. 57

The displacement has to be measured or annulled by some method in order that the range may be determined; an error in measuring the displacement corresponds to a proportional error in the determination of a base angle.

In Clerk's range-finder (1858), fig. 56 (*a*), the relative displacement of the images was estimated with the aid of an eyepiece micrometer, but it was found better to employ a null method and to bring the two halves of the image into exact coincidence or alignment by some suitable means. In Christie's range-finder (1886), fig. 56 (*c*), coincidence was secured by displacing one of the object glasses in a direction at right angles to its optical axis by means of a micrometer screw; in fig. 56 (*b*) a part of the optical system can be rotated by a suitable micrometer tangent screw.

The method employed in the Barr and Stroud range-finder shown in fig. 56 (*d*) is to produce a displacement of one part of the image by means of a laterally moving prism of small angle, the action of which will be understood by reference to fig. 57.

Let the angular deviating power of the prism $= d$, then the displacement of the image for a lateral movement L of the prism is equal to L*d*. As an object approaches from an indefinitely great distance to the minimum range which is required to be measured, the parallax angle (or difference of the angle *a* from 90°, fig. 55) increases from zero to $\frac{AB}{OA}$, taking the sine as the equivalent of the angle, since OA will be very large with respect to AB even at the shortest range. With an objective of focal length F this produces a linear displacement Δ of the image where

$$\Delta = F \, . \, \frac{AB}{R} \text{ (where } R = OA).$$

This must be annulled by the lateral movement of the prism, hence

$$d = \frac{F.AB}{L.R}.$$

This equation expresses the necessary deviating power of the prism in terms of the dimensions of the instrument and the minimum range required on the scale.

In the Zeiss range-finder the deflection is varied by two oppositely rotating thin prisms, the mounts of which are geared together so as to produce deviations of a central ray in one plane only. The two prisms are placed in front of one of the object glasses.

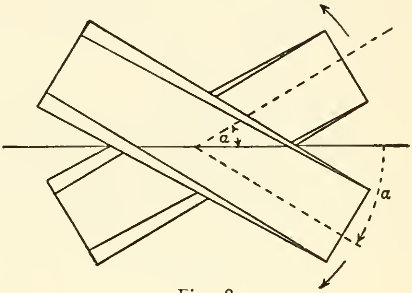


Fig. 58

Let the two prisms produce a maximum deviation $2d$ in the same sense; they are then base to base and edge to edge. If they are then turned through equal angles α (fig. 58) in opposite senses the deviation then becomes $2d \cos\alpha$. Hence the *change* in deviation produced by the movement $= 2d(1 - \cos\alpha)$. The total possible relative deviation is thus $4d$, and this must account at least for the whole parallax angle, although in practical instruments the rotation employed is not so large as 180°. In the actual instrument the prisms are cut to a circular shape for insertion in the mounts.

Division of the Scales.—In the case of the sliding prism the following equation was obtained:

$$d = \frac{F.AB}{L.R}.$$

Hence
$$R = \frac{F.AB}{L.d},$$

and
$$\frac{dR}{dL} = -\frac{F.AB}{L^2.d} = -\frac{R^2.d}{F.AB}. \qquad \ldots\ldots\ldots\ldots (2)$$

It is seen that the scale will be comparatively open at short ranges, but grows rapidly crowded at the longer ranges. In the case of rotating prisms we may make

$$2d(1 - \cos\alpha) = \frac{AB}{R},$$

or

$$R = \frac{AB}{2d(1 - \cos\alpha)}.$$

Hence $$\frac{\partial R}{\partial \alpha} = \frac{-\,AB.2d\sin\alpha}{\{2d(1 - \cos\alpha)\}^2} = -\frac{R^2}{AB}\,.\,2d\sin\alpha\,\ldots\ldots(3)$$

The range scale is divided on a ring which is geared to the prisms and moves through 360° for a 180° movement of the prisms. As before, the scale tends to be close towards the longer ranges, but something can be done in the above case by making α small when R is very large, so that the divisions of the scale are more open.

Limits of Accuracy of Range-finders.—On calculating the parallax angles (AOB, fig. 55) for various ranges they are found to be extremely small; for example, with a range-finder of base $1\frac{1}{2}$ yd. the angles at one or two distances are:

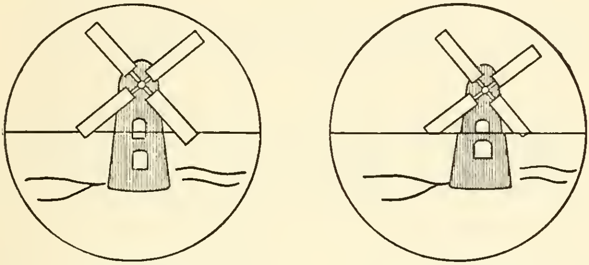
250 yards	21 minutes approximately
500 ,, ...	10 ,,
1,000 ,, ...	5 ,,
10,000 ,, ...	0·5 ,,

Suppose that it is required to attain a 1 per cent accuracy at 10,000 yd. under the above conditions. Substituting in equation (1) we may put $d\text{OA} = 100$, whence

$$d\alpha = \frac{1\frac{1}{2}.100}{(10{,}000)^2} = \cdot0000015 \text{ (angular measure)},$$

or about 0·3″ of arc. The combination of instrument and eye must therefore be capable of detecting this amount of angular parallax. It may be supposed that the range-finder is to sight on a black vertical line on a white ground (or vice versa, fig. 65, in the centre), and that the adjustment consists in bringing into exact alignment the separated halves seen in the two parts of the field. It is found that the sensitiveness of the eye to adjustments of this kind is much greater than the order of " resolving power " for separate objects. The eye cannot distinguish point objects, such as stars, which are separated by less than about 1′ of arc, but in the above case the residual separation of the edges after adjustment by average skilled observers corresponds to an angular subtense considerably less than 1′, being, in fact, 3″ to 10″ of arc after a little practice, or even less in special cases. Assuming a possibility of a mean error of less than 4″, it appears that with a magnifying power in the telescopic system of the range-finder of about ×14 it will just be possible to work to 1 per cent at 10,000 yards for a skilled observer, with the instrument of $1\frac{1}{2}$ yd. base, under the very best conditions.

Optical Arrangements for Halving the Field.—It has been suggested in a previous paragraph that a convenient object to " set " upon when the range-finder is used in a horizontal position is a vertical line. If this is viewed by means of the range-finder, the upper and lower parts of the image will appear out of alignment till the instrument has been correctly adjusted for the range of the mark. In some objects of which the range may be required no such convenient vertical line will be found, and it is necessary to effect " coincidence " between the parts of images of irregular objects. It is then essential that the " halving " of the field shall be exact along the line of separation. This means that if a man were seen in one part of the field of view and the dividing line were found to cross the centre of a button in his coat, the other half of the image, whether inverted or erect, must start exactly from the centre of that button. The defect of incorrect halving will readily be appreciated by reference to fig. 59. The method of correcting the halving of the field will be indicated in a later paragraph.



Correct Halving

Incorrect Halving

Fig. 59

There are a number of different methods of dividing the field. In the simplest coincidence type of instrument the division is into equal upper and lower halves, both parts of the image being erect.

There are also the "erect band" and the "inverted" systems, the arrangement in which will be understood by reference to fig. 60. In the majority of cases the division is secured by composite prisms of special design which are somewhat difficult to describe briefly on paper. The main action of the centre prism system of one of the Zeiss range-finders can, however, be followed by reference to fig. 61. The face AB is silvered except for a small opening O which is arranged to be in the focal plane of the instrument.* Light from the two object glasses enters from the left and right respectively, that from the left suffering a triple reflection in the first prism. The light from the right is reflected upwards by the lower "roof" prism, and of all this light the rays which pass through the opening O in the silvered separating surface are the only ones to reach the eye. On the other hand the left-hand light supplies an image which is complete except for the band O, as, owing to the fact that the

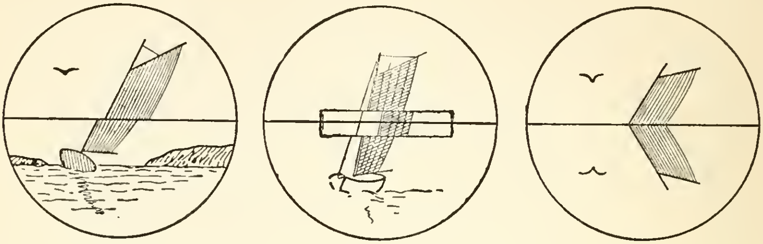

(1) (2) (3)

Fig. 60—Division of Range-finder Field

1, Coincidence system. 2, Erect-band system. 3, Inverted-image system.

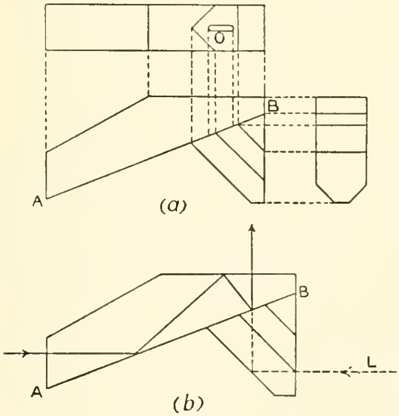


Fig. 61

(*a*) Elevation, Plan, and End View of the Separating Prism of a Zeiss Range-finder.
(*b*) Ray Paths in Zeiss Prism System.

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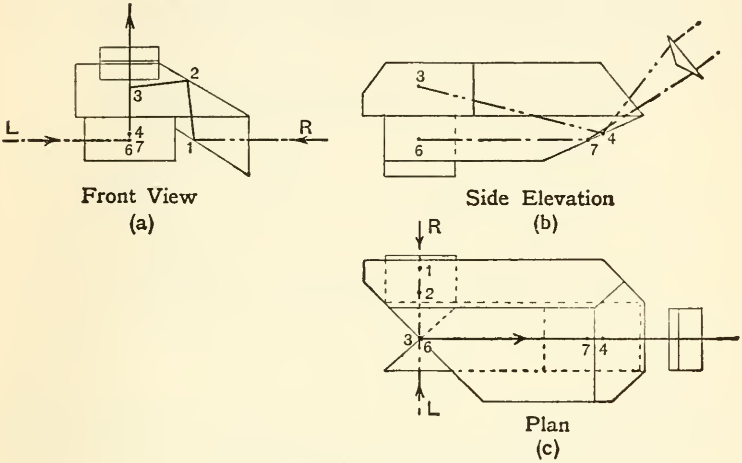
* If the centre of the band is in the focal plane, the lateral parts of the separating edges will not be in clear focus, as the aperture is not perpendicular to the optic axis. The appearance is suggested in fig. 60 (2).

two parts of the prism are balsamed together, a part of the light from the left passes through the opening in the silver without reflection and is not seen. With this arrangement, when a distant object is being viewed and the direction of sighting is horizontal, the observer must look into the eyepiece of the instrument in a horizontal direction also. Some types of range-finder, due to Messrs. Barr & Stroud, have a special addition to the centre separating system by means of which the emergent rays may be thrown upward, thus enabling a more convenient and restful position of the head to be employed for observation.



Front View (a)

Side Elevation (b)

Plan (c)

Fig. 62.—Barr & Stroud Centre Prism

Fig. 62 shows diagrammatic views of one of the centre separating prism systems due to Messrs. Barr & Stroud. From the viewpoint of the observer the light from the right is reflected at 1 (fig. 62 (*a*)) in an upward direction, then slightly downward at 2. Referring now to the plan, the ray is seen to be reflected through a right angle in azimuth at 3, but it maintains a downward tendency and is finally reflected upward at 4 and thence passes to the biprism shown in section in the side elevation. The left-hand ray suffers a right-angle reflection (at 6) at a lower 45° face immediately on entering the prism. It then passes forward and is reflected upward to the biprism at 7.

The refracting edge of the biprism is in the common focal plane of the objectives of the instrument. The right-hand beam thus illuminates the upper portion of the field while the left-hand beam produces the lower portion. The arrangement as thus described will produce a simple coincidence type of division of the field. It secures the advantage of the convenient downward direction of view.

Eyepiece.—The eyepiece employed is usually of the "Ramsden" or positive type, and is provided with a focusing lever. It is usually the case that one eye alone observes the field of view, while the other views the scale, which can be illuminated at will, with the aid of a separate eyepiece or magnifier.

Method of Construction.—The development of the construction, both optical and mechanical, of the range-finder has been along the lines prescribed by its main purpose as a military instrument and the consequent rough usage which it is likely to suffer. In one Turkish test the instruments were dropped from a height of 2 ft. across the trunk of a tree; if then the indications of an instrument were rendered inaccurate it was rejected. Not all range-finders are called upon to undergo trials of such severity, but in general the outer case of the instrument consists of a tube of steel suitably protected by outer coatings of leather, canvas, &c. Portable instruments have ends protected by rubber ringed discs of larger diameter than the tube.

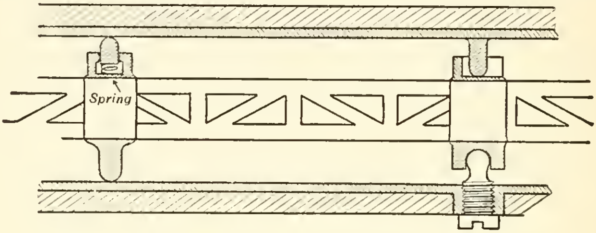
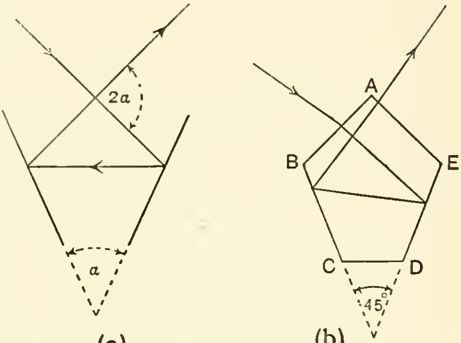


Fig. 63.—Mounting for Interior Framework (diagrammatic)

The interior optical system, comprising the object glasses, the deflecting system, and the centre separating prism system, are mounted together in a separate girder framework. This framework has to be held in the outer tube in such a way that distortion or strain of the outer tube is not communicated to the girder. In one type of range-finder (fig. 63) this condition is secured as follows. The girder framework, which is of square section, is encircled near its ends with two stout bands. The first of these bands is furnished with three projections, one of these being a spring pin which secures that all three projections remain in contact with the wall of the outer tube. The other band carries on its lower side a cup and on its upper side a longitudinal groove; these engage respectively with (1) the head of a screw passed through the outer tube, and (2) a projection made for the purpose on the opposite side of the tube. This mounting is sufficient to secure, on more or less geometric principles, stability of the interior fitting and avoidance of the possibility of bending it by small strains on the outer case of the instrument.

The relative stability of the interior optical parts from the object glasses to the centre prisms would avail little if displacements could be given to the images by the small angular shifts of the end reflectors, which would be the result of strain on the outer tube, for it is usual to place the end reflectors on special mounts fitted into the ends of the outer case. Such reflectors must therefore be used which, when the range-finder is directed horizontally, could be given small rotations in azimuth without producing corresponding deflections of the image. Now the deviation produced when a ray is reflected successively by two mirrors held at a fixed angle to each other is independent of the angle of incidence of the ray on the first reflector (provided that the ray and two normals to the reflectors are found in the same plane). The amount of the deviation is always double the angle between the reflectors; thus to produce a deviation of 90°, reflectors at 45° must be employed (see fig. 64 (*a*)).



(a) (b)

Fig. 64

For the very largest types of instrument (large naval range-finders) the two reflectors (in some cases silvered glass mirrors with optically worked surfaces, and in other cases metallic surfaces) are held at a fixed angle by means of gun-metal castings, but in the smaller instruments with bases 1 or 2 yards in length it is found more convenient to employ glass prisms of the type illustrated by ABCDE in fig. 64 (*b*) and known usually as "pentagonal" prisms. The faces BC and DE are the essential reflectors, and, as before, are inclined at an angle of 45°. The path of a ray through the prism is sketched. Since reflection at BC and DE has to take place at angles of incidence considerably less than the critical angle, these faces have to be silvered. The faces AB and AE are inclined at 90°, and are thus normal to rays passing symmetrically through the prisms. The action of the prism is thus substantially the same as a block of glass with parallel faces in addition to the reflectors.

Such prisms or built-up reflectors will cause no deviation of the images when suffering small angular rotations in azimuth, especially those exceedingly slight rotations which will be likely to be produced by the slight distortion of the external tube due to temperature effects and other various causes.

Adjustments of the Range-finder.—The adjustment of the coincidence type will be shortly described. A convenient test object to set upon for adjustments is the full moon. Should this be available the instrument is directed upon it, and the milled head which operates the deflecting system is rotated until the two half images are as nearly in coincidence as possible. The " halving " is now corrected by means of another milled head at one end of the instrument, which, by means of a pinion and screw, tilts one of the end reflectors so as slightly to displace the image in one half of the field in a direction perpendicular to the line of separation. The halving is also adjusted in some instruments by the displacement of one of the objectives or by the tilting of an auxiliary plane parallel plate of glass between one objective and the image. The range scale should read " infinity " when correct halving and coincidence have been effected as perfectly as possible.* If it does not do so the necessary correction may be made by the help of a prism of very small angle, which is often mounted as a window in front of one of the end reflectors. It is provided with means for giving it rotation in its mount. In the ordinary position the refracting edge of the prism is likely to be horizontal when the range-finder sights in a horizontal direction. Hence any rotation will introduce small deflections of one beam in azimuth, and the range indications may be corrected till there is perfect coincidence between the images of the moon when the range scale reads infinity.

Should the moon not be available, an alternative is to sight upon the " lath adjuster " often provided by the makers. This may conveniently be a support carrying two similar discs about 4 in. in diameter, the centres of which are at a distance apart equal to the characteristic base length of the instrument. The support is fixed at some distance, say 100 yd. or more, so that its length is parallel to the range-finder. The image of the left-hand disc formed by the left-hand optical system should coincide with the image of the right-hand disc formed by the right-hand system when the scale reading is infinity (fig. 65). A short bar of white metal on the diameter of each black disc is the object on which the real setting is to be made.

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* The mean of several observations is required in adjustment as also in range-taking.

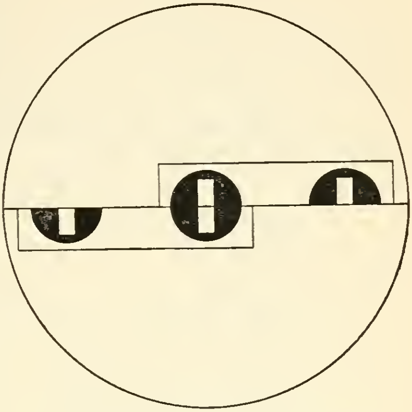
Fig. 65.—Lath Adjuster seen in Field of Range-finder (diagrammatic)

There are also certain types of self-contained adjusting systems which can be fitted to range-finders, and reference will be made to these after the discussion of the stereoscopic range-finder.

Stereoscopic Range-finder.—The subject of stereoscopic vision is one which it will not be possible to discuss adequately in the present account,* and the psychological phenomena of this type have not yet been fully explained, but we can summarize the main points as follows.

A person observes two objects A and B distant, say, 100 yd. and 200 yd. respectively. They are depicted on the retina of each eye with equally satisfactory definition, but owing to the separation of the view-points of the two eyes (x and y, fig. 66) the distances between the images of A and B on the retina is not the same in the two cases.

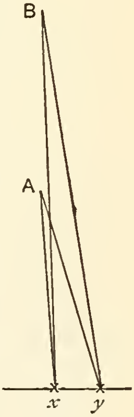

Fig. 66

It is a matter for speculation as to the precise nature of the mechanism by which the single mental image is formed, with its sensation of the stereoscopic depth or distance displacement between the two objects, for there is often no apparent doubling of either of the images. Granted that the visual axes of both eyes are directed to the one point A, then the angle subtended between A and B will be different to the two eyes. This difference is known as the "parallactic displacement"; when it exceeds a certain value there will be a doubling of the image of B. From another point of view it may be said that a difference of convergence is needed when the eyes view one object or the other. This "convergence" may be partly mental; there may be a difference (with steady vision) in the mental efforts required to associate the two images in the cases of A and B respectively. On the other hand, the actual convergence of the visual axes does change when viewing points at very different distances, and it is at all events probable that the full sense of the solidity of objects depends partly on the free "play" of the eyes over them, together with the supplementary psychological effect. From the mathematical point of view it is justifiable to associate the stereoscopic sense of relief with *differences* of angular convergence, although not with the absolute value of the angle. We have very little, if any, power of judging the absolute distances of objects through the stereoscopic sense except by reference to other modes of psychological experience. Thus when the term "convergence" is employed in the subsequent discussion its significance must be interpreted in the light of the foregoing remarks. From the point of view of the present writer the conception of convergence seems the one most easy to employ when discussing the stereoscopic range-finder.

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* Reference may be made to *Neue Stereoskopische Methoden und Apparate*, Pulfrich (Berlin, 1909), also to *Stereoskopische Sehen und Messen*, by the same author.

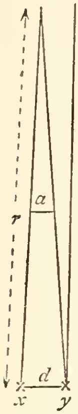

Fig. 67

Beyond a certain distance, however (about 450 m.), no stereoscopic effect is perceptible. Let r_0 be the distance of an object which can just be distinguished as being nearer than one at a very much greater distance (several miles, say).

The directions of the two eyes may be taken as parallel when looking to an object at an indefinitely great distance, but when looking to an object at a distance r these directions are inclined at an angle a (fig. 67).

Then if d_0 is the distance between the centres of rotation of the eyes,

$$a = \frac{d_0}{r},$$

and for the maximum range of stereoscopic vision

$$a_0 = \frac{d_0}{r_0},$$

where a_0 is the least appreciable angle of convergence. One experimentally determined value for r_0 is 447 m. Taking d_0, the interocular distance, as 6·5 cm., this gives $a_0 = \cdot 000145$, or about 30″ of arc. It will be expected that this least appreciable angle of convergence will be significant in determining the least *change* in convergence which will serve to distinguish a difference in the distance of two objects in the visual field of view.

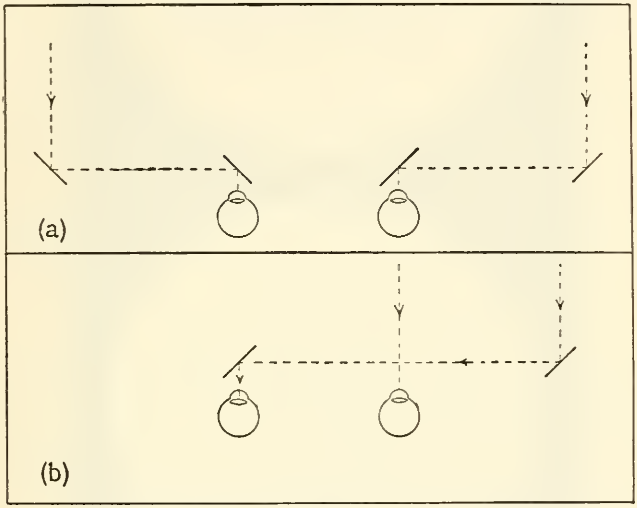

Fig. 68.—(*a*) Augmented Relief. (*b*) Inverted Relief.

By artificially increasing the distance apart of the view-points a sensation of increased relief may be obtained, or by artificially reversing the usual view-points we may produce a sensation of inverted relief. Means of doing this with small mirrors mounted on a bar are suggested in figs. 68 (*a*) and (*b*).

If a telescopic system of magnification M is used in conjunction with a system similar to that shown in fig. 68 (*a*), then all angles are magnified to the eye, and the limiting angle of resolution becomes $\frac{1}{M}$th part of its original value on account of the magnification alone.

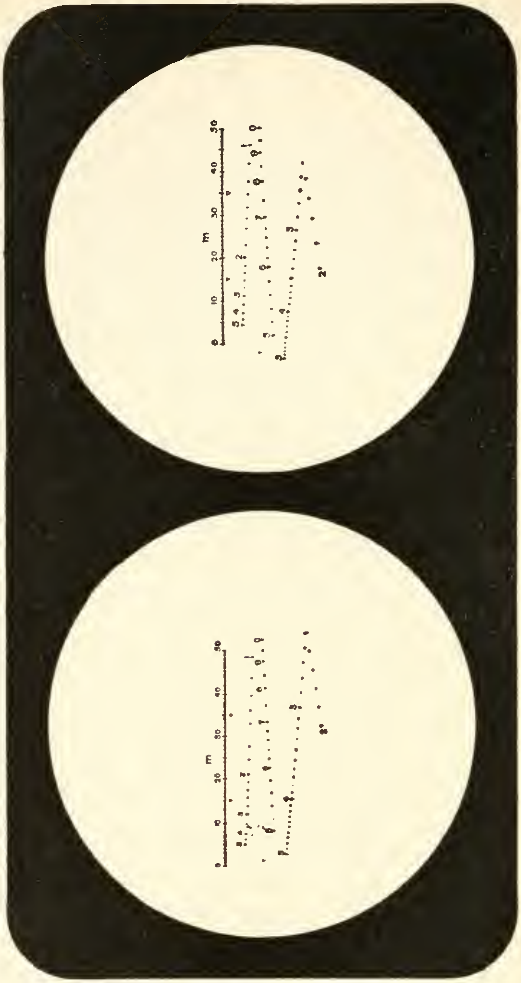

Fig. 69.—Scale to show Relief

Also with an increased separation d of the view-points and a magnification M, the new stereoscopic radius becomes

$$R = r_0 \cdot \frac{d}{d_0} \cdot M.$$

The product $\frac{d}{d_0}.M$ is known as the *total relief* or *stereo power* and is often denoted by Π.

It will be seen that if we simply increase magnification without the use of a greater base we bring objects apparently to $\frac{1}{M}$th of their original distance, when they will be subject to increased stereoscopic effect.

Increasing the distance between the view-points or the "base distance" without increasing the magnification also brings objects apparently nearer, although they still possess the same angular subtense. The appearance is therefore that of a nearer *model* of the objects in the field of view.

Most persons are familiar with the ordinary stereoscope. This may be regarded as consisting of two eyepieces or magnifiers viewing respectively two pictures which exhibit, when viewed, effects of relief associated with the differences of convergence necessary in viewing or "focusing" the images of the different objects represented in the pictures.

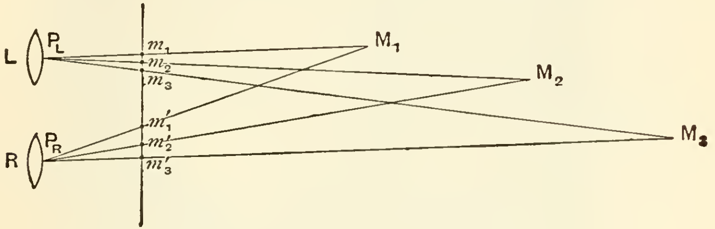

Fig. 70.—Stereoscopic Projection of Points

It is easy to make up line pictures which when so viewed will exhibit stereoscopic relief, and one may represent a scale stretching out into space with various distances marked upon it. Fig. 69 shows two drawings to give such an effect* in a stereoscope. The slight displacement of the lines relatively to the circle in the two cases will be rendered apparent by taking a tracing of one and applying it to the other. The reason of the effect may be further elucidated by reference to fig. 70. L and R are lenses representing the left and right eyepieces of the stereoscope. In the focal plane of these lenses are found marks m_1, m_2, m_3 and m_1', m_2', m_3'. Images of these marks are thus seen projected by the lenses to a much greater distance. Taking P_L and P_R as the corresponding principal points of the lenses, it will be seen that the eyes may be converged to look along the directions $P_L m_1$ and $P_R m_1'$ respectively, when the combined image of the two marks will be seen apparently situated at some point M_1 in the space-image presented to consciousness. M_2, which requires a smaller angle of convergence, mental or physical, will appear at such a distance beyond M_1 as would call for the same change of convergence, and similarly for M_3. M_3 might be any point which the mind consciously or unconsciously assigns to some known distance, or, say, an indefinitely great distance. The positions of M_2 and M_1 will then be assigned with respect to M_3.

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* Many persons can fuse the pictures by "squinting" outwards when the eyes are accommodated to view them.

Fig. 71 represents a stereoscopic instrument made by combining the mirror system of fig. 68 (*a*) with two telescopes; prisms are used in place of ordinary mirrors. To convert it into a range-finder it is simply necessary to place transparent graticules with suitably engraved lines in the focal planes of the eyepieces so as to produce, as before, a stereoscopic picture of a scale stretching out into space. This will be seen at the same time as the stereoscopic view of the field, and distances of objects may be judged with reference to it. The appearance is suggested in fig. 72, which must be viewed by a stereoscope or by " squinting ". In the actual instruments a system of erecting prisms is employed so that an upright image is seen.

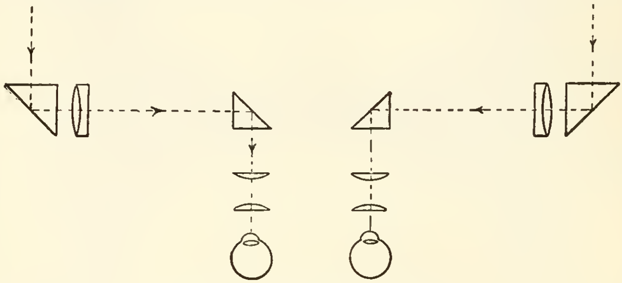

Fig. 71.—Stereoscopic Telescope

There are a number of more or less successful variations of this stereoscopic principle. In one case, the " wandering mark " range-finder, the graticules bear one mark only and the fused image is made apparently to recede or approach by moving one of the graticules. A very sensitive gearing arrangement is necessary so that ranges may be read off from a scale engraved on a member of the gearing system with a comparatively large motion.

Accuracy of Stereoscopic Range-finder.—The ability to use a stereoscopic range-finder is largely a matter of individual

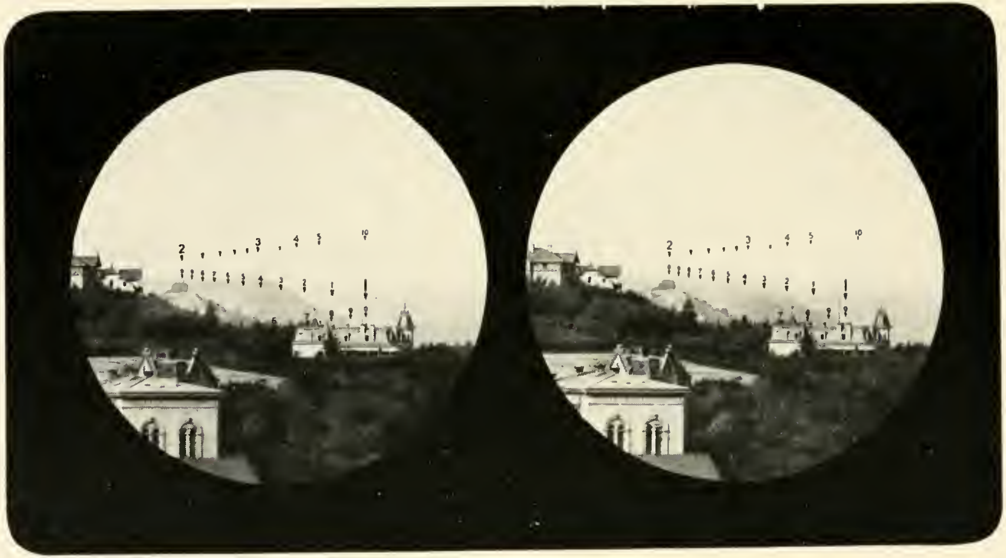

Fig. 72.—Stereoscopic View of Scale in Range-finder

training, as the stereoscopic sense varies very widely between various persons. Pulfrich devised a special stereoscopic slide for testing ability in this connection. With the best observers, however, an order of accuracy which approaches that of the coincidence type of range-finders (with equal base length) can be obtained, but it is said that the coincidence type still maintains a definite lead in accuracy where deliberate readings are to be taken.

The stereoscopic instrument possesses certain advantages, in quickness of reading and the absence of moving parts, which are very valuable. It was invented by H. de Grousilliers in 1892.

In a recently published book * Pulfrich again draws attention to the several advantages of stereoscopic range-finders, which were largely used in the German navy during the War. Not only can the distance of the mast of a ship be estimated, but also that of such things as smoke clouds, or even shell bursts and splashes, on which it would be clearly impossible to use range-finders of the coincidence type.

If it is so desired, another important addition may be made to the stereoscopic range-finder. Fig. 72 shows a drawing of a simple scale, but each scale division might easily be made to represent a vertical " foot rule " (say) seen under the visual angle which it would subtend at the corresponding distance. While a foot rule would, of course, be too short, it will be understood that it is easily possible *apparently* to project " scales " (held apparently at right angles to the line of sight) into the field of view, to come into coincidence with the objects seen at the same distance in the space picture. Thus not only can the distance of these objects be estimated by the power of stereoscopic vision, but their size also. Instruments of this kind are of the greatest possible value for many purposes, such as, for example, the estimation of the height of waves on the sea.

Internal Adjusting Systems for Range-finders.—All the larger range-finders, especially the naval types, now carry a self-contained means of " infinity " adjustment which can be put into action when desired. The optical principles of two of these may be briefly indicated.

(*a*) *The Abbe Adjuster.*—This is applicable to stereoscopic range-finders and may be understood by reference to fig. 73 (*a*). It makes use of the convenient constant right-angle deflection produced in a beam of light by means of the " pentagonal " prism.

Light is derived from the source L and illuminates the graticule in the right-hand side. A ray from the division marked infinity passes on out of the instrument and is brought back into the left-hand side by two successive right-angle reflections in suitably placed " pentagonal " prisms. A state of affairs similar to that when the range-finder sights on an infinitely distant object and thus receives two parallel beams of light has therefore been produced. The image of the infinity mark in the right-hand graticule should therefore be formed in coincidence with the infinity mark in the left-hand graticule. The accuracy of the test is completely dependent on the performance of the prisms, which themselves must be tested before use. The deviation will not be exactly 90° unless the angles of the prisms are correct, but it would be easy to introduce compensating prisms to annul any error in the adjuster.

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* C. Pulfrich, *Die Stereoskopie im Dienste der Photometrie, etc.* (Berlin, 1923).

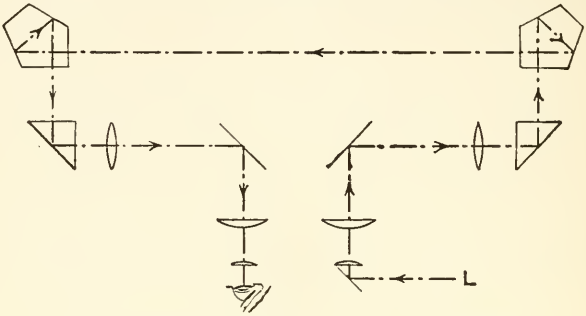


Fig. 73*a*.—Abbe Adjuster for Stereoscopic Range-finder

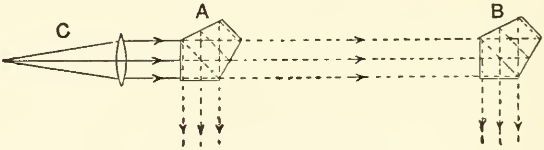


Fig. 73*b*.—The König Adjuster

(*b*) *The König Adjuster.*—König has patented an arrangement of two pentagonal prisms and a collimator by means of which two accurately parallel beams may be produced. If this is possible the apparent range of the mark in the focus of the collimator can be taken by the range-finder. If the beams are in reality accurately parallel the range of the mark should be " infinity ". The parallel beam from the collimator C (fig. 73 (*b*)) passes half over and half into the pentagonal prism A. The undeviated portion passes into B.

The two parallel beams then enter the windows of the range-finder. If a coincidence range-finder is being tested, the images of the collimator mark should coincide when the instrument scale reads infinity.

In the Zeiss long-base range-finders a similar method of adjustment is employed, but the observation is taken with a collimator at opposite ends in turn.

A Double-image Range-finder.—Mr. Dennis Taylor, of Messrs. Cooke, Troughton, and Simms, Ltd., has designed a range-finder, the principle of which is illustrated by reference to fig. 73 (*b*), imagining the path of the light to be reversed. Two nearly parallel beams, derived from a distant object, enter the two pentagonal prisms placed at a distance apart which is determined by the required base length. The reflected beams then enter the top and bottom halves of a telescope object glass, and the image of a comparatively near object will thus be doubled in the field of view of the telescope until the doubling is eliminated by the action of a prism of small angle placed in the path of one of the beams. In practice two prisms are used, one in each beam. Rotation of one of these prisms about an axis parallel to the refracting edge allows small changes of deviation. The arrangement has the advantage of using one object glass only; thus the instrument does not depend on the rigidity of the connection between two lenses.

CHAPTER VIII

Spectrometers, Angle Comparators, and Auto-collimators

The Spectrometer

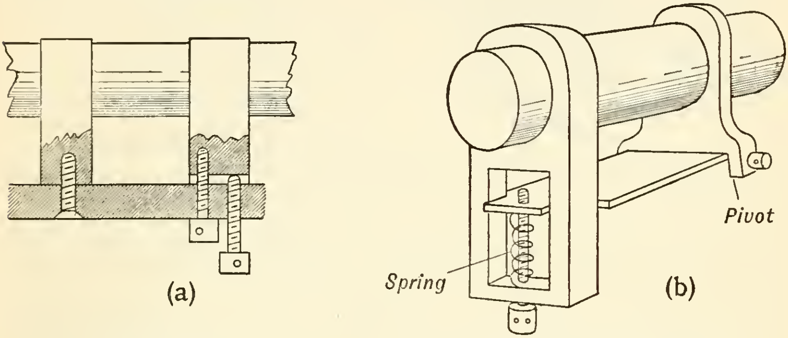
The spectrometer is an instrument of fundamental importance for the measurement of refractive index, and the indications of other refractometric instruments generally depend on spectrometer results. It is the only instrument for glass refractometry of the highest accuracy.

Parallel light is produced by a collimator and directed over the axis of the instrument, where it may be altered in direction by refraction through a prism. The light is received by a sighting telescope pivoted on the central vertical axis of the spectrometer. The angular movement of the telescope is measured with the aid of verniers or micrometer microscopes moving over a divided circle. A " table " for the support of the prism is also mounted to rotate about the central axis, and this may also carry verniers moving against a second circle. The table is usually provided with means for levelling.

In some cases the circle has an independent mounting, and may be rotated relatively to the collimator. The advantages of this have already been discussed (p. 58).

Mechanical Construction.—The chief new points (other than those already dealt with in regard to the theodolite) needing attention in regard to the mechanical construction are concerned with the mounting of telescope and collimator. The centre of gravity of the telescope is a long way from the axis, so that a counterbalancing weight is necessary. In some instruments, however, the weight of the telescope is borne by a circular slide. An extra long axis for the moving parts is an advantage in the first case. The remarks made in Chapter IV with regard to verniers and the divided circle apply of course to the present subject.

The supports for telescope and collimator must permit of adjustment while retaining sufficient rigidity. In many school spectrometers the levelling of the optical axis is intended to be done by means of capstan screws, the use of which, however, distorts the tube (fig. 74 (*a*)). A better method of mounting is shown in fig. 74 (*b*). For ordinary goniometric purposes it is unnecessary to provide independent rotations on their mounts for the collimator and telescope, but it may be an advantage if it is desired to use the instrument at any time as a spectroscope with several prisms, in which case the incident and emergent beams are no longer directed to and from the central axis.



(*a*) Levelling by Distortion of the Tube

(*b*) Better Method of Adjustment

Fig. 74

The table is supported by three levelling screws and must be adjustable in height relatively to the optical parts, unless the height of *these* is adjustable. Clamps and fine adjustments are always provided for table and telescope.

The slit is one of the most important of the mechanical parts, and the opposing jaws are ground straight and fine. Care must be taken in their mounting so that the slit remains of a relatively uniform width even when it is extremely fine. The jaws must be in the same plane. Fig. 75 shows two constructions for a slit. There is no special advantage for ordinary spectrometer work in making both sides movable. The most usual plan is to move one of the jaws by a micrometer screw opposed by a spring (fig. 75 (*a*)). In this case the head of the screw may be graduated to register the width of the slit.

Optical Design.—Parallel light must be produced by the collimator and brought to a sharp central focus by the telescope. Spherical correction of the lenses is thus very important, and it would be desirable to give this for two wave-lengths; but there are reasons against this (p. 18), and the most common practice is to employ ordinary doublet telescope objectives. The presence of the secondary spectrum sometimes renders necessary (when the greatest accuracy is required) a separate collimation for each wave of light employed. Small amounts of central astigmatism are by no means a disadvantage if the lenses can be turned, and may be used to produce line images of each point of the slit, thereby hiding the effects of dust and small irregularities. This expedient is, however, more useful in spectrographs (photographic instruments) for which the optical requirements are somewhat different. In the latter case unachromatic single lenses, suitably figured for the reduction of spherical aberration, may be employed. The plane of the plate has to pass through the locus of the foci for the range of wave-lengths employed, and slight rotundity of this field may be allowed for by bending the plate. In some cases photographic attachments may be fitted to ordinary spectrometers.

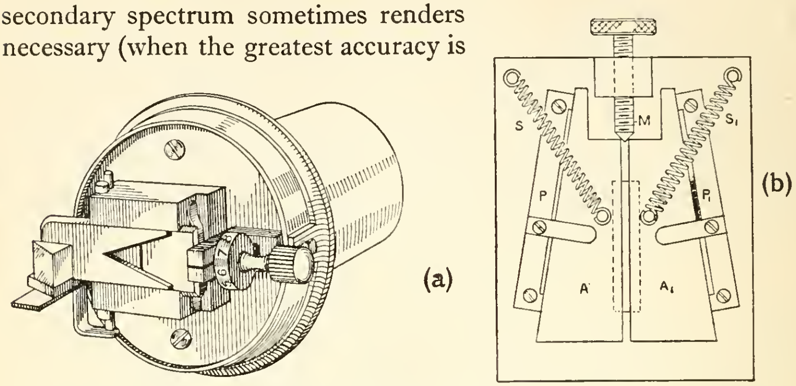


Fig. 75.—Slits for Spectrometer

(*a*) Slit designed by Messrs. Adam Hilger, Ltd. (*b*) Slit described by Whipple (symmetrical type). A, A_1, Jaws. P, P_1, Slides. S, S_1, Springs. M, Screw to control aperture.

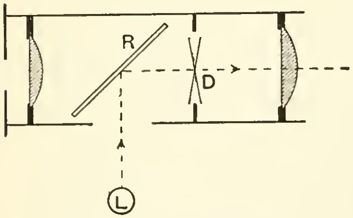


Fig. 76.—Gauss Eyepiece
L, Source of light. R, Plain glass reflector. D, Diaphragm carrying cross wires.

A Gauss eyepiece (fig. 76) is a useful *addition* to the ordinary equipment. It consists of a Huygenian eyepiece with an aperture and internal reflector (of plane transparent glass) for the illumination of the cross wires. The light is directed through the field lens of the eyepiece towards the objective.

Adjustment.—The chief adjustments are:

1. Setting the axes of telescope and collimator perpendicular to the axis of rotation.
2. Collimation or adjustment of the telescope to receive parallel light and the collimator to yield such.

There is no need to repeat instructions for the various methods, which are given at great length in every textbook of practical physics, but two good methods may be briefly indicated. For No. 1 a piece of plate glass with parallel faces is mounted vertically on the table, and so adjusted that an image of the cross wires, formed by light reflected from the face of the plate glass, is found central in the field of view of the telescope carrying the Gauss eyepiece, with a suitable illuminant in position. The telescope is then rotated through 180°. If its axis is perpendicular to the axis of rotation, the image on the other side will also be perfectly central; but if not, half the error may be corrected by adjusting the telescope and half by moving the glass with the levelling screws of the table. The process is then repeated until the adjustment is perfect. The reason of the procedure is obvious.

For No. 2, having given a really good 60° prism, the method usually known as Schuster's is very convenient. For angles of deviation somewhat greater than the minimum, *two* positions of a prism may be found which will give the deviation required to bring the image into the field of the telescope. If the light incident on the prism be truly " parallel ", the image will be in perfect focus in the two positions of the prism. If not, the divergence of the beam after passing through the prism is considerably different in the two positions. The prism is turned to that position in which the angle of incidence of the light on the prism face is greater than that for minimum deviation and the *telescope* is adjusted to focus the image. The prism is turned to the other position and the *collimator* is adjusted to focus, then back, and so on till there is perfect focus in both positions of the prism. If the wrong order is adopted the focus rapidly becomes worse instead of better.

If a prism is to be used it is previously adjusted so that its " refracting edge " is parallel to the axis of the instrument by the aid of the levelling screws of the table, in the manner usually described.

Measurement of Refractive Index.—The usual method is to determine the angle A of a prism and the amount of the minimum deviation D for the spectrum line employed. The refractive index is then calculated by the formula:

$$N = \frac{\sin\frac{A + D}{2}}{\sin\frac{A}{2}}.$$

Differentiating the formula with regard to A, and substituting approximate values for the angles such as are likely to hold in actual experiments ($A = 60°$, $D = 38°$, or nearly), we get $dN = 0{\cdot}7\, dA$. From this it is easy to show that an error of 1′ of arc in measuring the prism angles corresponds to an error of 0·0002 in the result for refractive index. An error in measuring the deviation produces about the same error in N as the prism angle inaccuracy.

Accuracy Obtainable.—Cheap quality spectrometers may give readings uncertain to 1′ or 2′ of arc, and the error in the refractive index determination may thus amount to 4 or 5 units in the fourth decimal place. With better instruments of a simple type (with verniers reading to 30″) the probable error may be reduced to about 1 or 2 units in the fourth place by taking the mean of several results. Finer instruments have verniers reading to 5″ or 10″ of arc, and, by careful work with these, refractive indices may be found to one or more units in the fifth decimal. In attempting still higher accuracy several points must be remembered.

Considerations necessary in " sixth-place " work.—(1) The variation of refractive index with temperature. The " temperature coefficients " for glass vary in sign and magnitude, but a change of 1° C. may affect the fifth decimal place for some glasses.* The effects of temperature, pressure, and humidity on the refractive index of air are also to be borne in mind.

(2) The methods employed for determination of the angles and deviations will have to be chosen with great care or error will result (see later paragraphs of this chapter).

(3) The errors of the divided circle, which are possibly of the order of a few seconds of arc, must be remembered. A test or calibration of the circle is imperative, or special precautions (p. 58) must be taken.

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* For particulars of temperature effects on glass see *Jena Glass* (Hovestadt), p. 57.

(4) If glass prisms are the subject of test it must be remembered that glass is not homogeneous. Local variations in composition may change the refractive index by several units in the fifth place. Glass intended for sixth-place work should be examined with an interferometer.

In our present state of knowledge it is very doubtful how far all these difficulties have been successfully overcome in any set of measurements, so that results given to the sixth decimal place have to be accepted with some reserve.

Determination of Prism Angles.—Two methods for the ordinary determination of prism angles are usually described in the textbooks. The first of these is a " split beam " method in which a parallel beam from the collimator should be divided into two parts, the angle between which is measured by the telescope and is twice the angle of the prism. If, however, the beam from the collimator should not be truly parallel, the mean rays on each side will not give correct determination of the prism angle. Furthermore it should be noted (referring to fig. 77 (*a*)) that, supposing C to be the collimator lens and t_1 the telescope lens in its first position, the image is being formed by the shaded parts of these lenses. When the telescope lens is at t_2 the unshaded parts will be in action. Should the curvatures of the lenses not be prefectly uniform, or should spherical aberration be in evidence, there may be uncertain effects leading to error. Incorrect focusing would also obviously produce error. This method should therefore be avoided when possible.

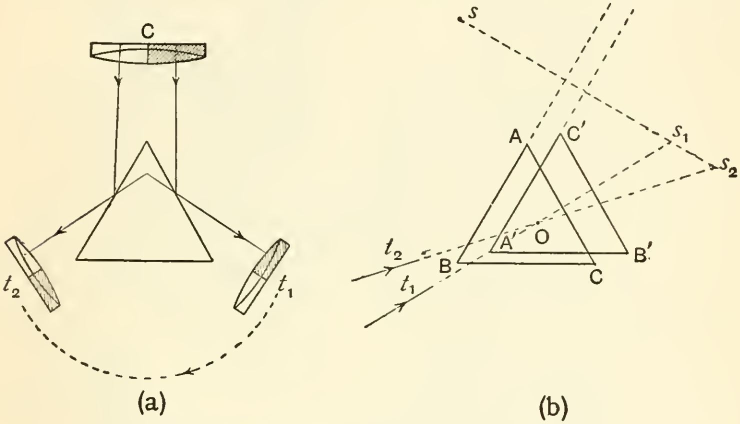

Fig. 77.—Determination of Prism Angles: Possible Errors

The alternative is to keep collimator and telescope relatively fixed while the prism is rotated through an angle of 180° — A, where A is the angle to be measured. In this case suppose the collimator to be wrongly adjusted so that the image of the slit is at a finite distance. Further, suppose that the prism is so situated on the table of the instrument that the two surfaces enclosing the angle which is to be measured are not equidistant from the centre of rotation. The condition is shown diagrammatically in fig. 77 (*b*). s is the image of the slit formed by the collimator lens, and s_1, s_2 are the two images made by reflection in the faces AB and A′C′ of the prism in two positions, such that in the second the face AC is brought parallel to the first direction of AB.

If s is at a finite distance s_1 and s_2 will be at a finite distance, and the angle between their directions will be finite. Hence the prism must be turned through a further angle in order that s_2 may be sighted along the first direction of the telescope, and an error in the determination is thus caused. The trouble is minimized by taking great care with the focusing and the placing of the prism, so that the conditions are, as far as possible, similar in each setting.

In some of the larger spectrometers there is no divided circle for the prism table. This does not mean that resort to the split-beam method is necessary. Gifford employs an equilateral prism of which the angles are equal to 60° as nearly as possible. The deviation is found for each angle and the mean is taken, the angle then being treated as exactly 60°. It is, however, necessary to remember the possibility of pyramidal error in the prism.

Chromatic Parallax.—In ordinary practice the chief errors arise in the setting of the image on the cross wire. In instruments like the theodolite the cross wire is silhouetted, and no difficulty arises similar to that experienced when a spectrometer is used in darkness and the slit is very fine. In this case it is difficult to see the cross wire at all. The usual expedients are either to use a very fine vertical cross wire (or " occulting wire ") and a slit somewhat wider than usual (setting the wire centrally in the image), or to introduce a certain amount of stray light into the field sufficient to make the cross wire visible. Serious apparent parallax* may occur if the cross wires are seen by light differing greatly in wave-length from the spectrum line in the field, and this may lead to very erroneous settings. Methods of overcoming the difficulty are suggested by Guild:

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* Guild, *Proc. Phys. Soc.*, Lond., Vol. XXIX, May, 1917.

1. The use of a fairly wide slit with a fine vertical spider line in the centre. Settings are made on the dark image of this line.

2. The addition of white light to the radiations forming the spectrum. A continuous spectrum which will silhouette the cross wires may thus be produced. The white light derived from a fairly powerful source may be concentrated by a lens and reflected on the slit by a thin piece of ordinary glass which will not obstruct the light, such as that from a hydrogen tube, which is being used to give a line spectrum.

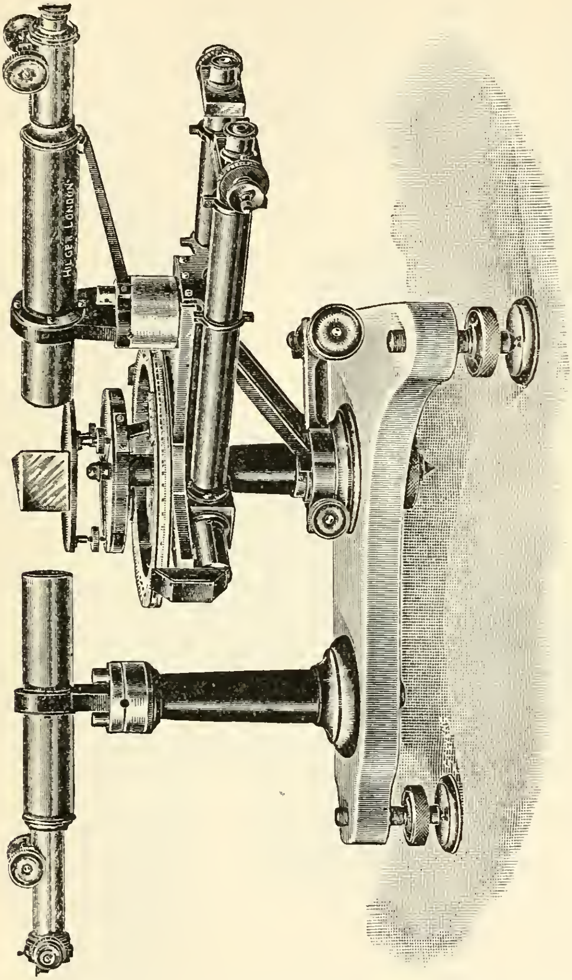

Fig. 78.—Spectrometer by Hilger

Light from the extreme ends of the slit reaches the image by paths considerably inclined to the optical axis. A greater deviation will be produced in these rays by the prisms, and consequently the spectrum lines will appear a little curved. This calls for the use of a very short slit if the method of the occulting wire is adopted. If a heliostat is available and sunlight can be employed, the Fraunhofer lines can be employed for refractometry. In this case the cross lines are well shown up against the continuous spectrum.

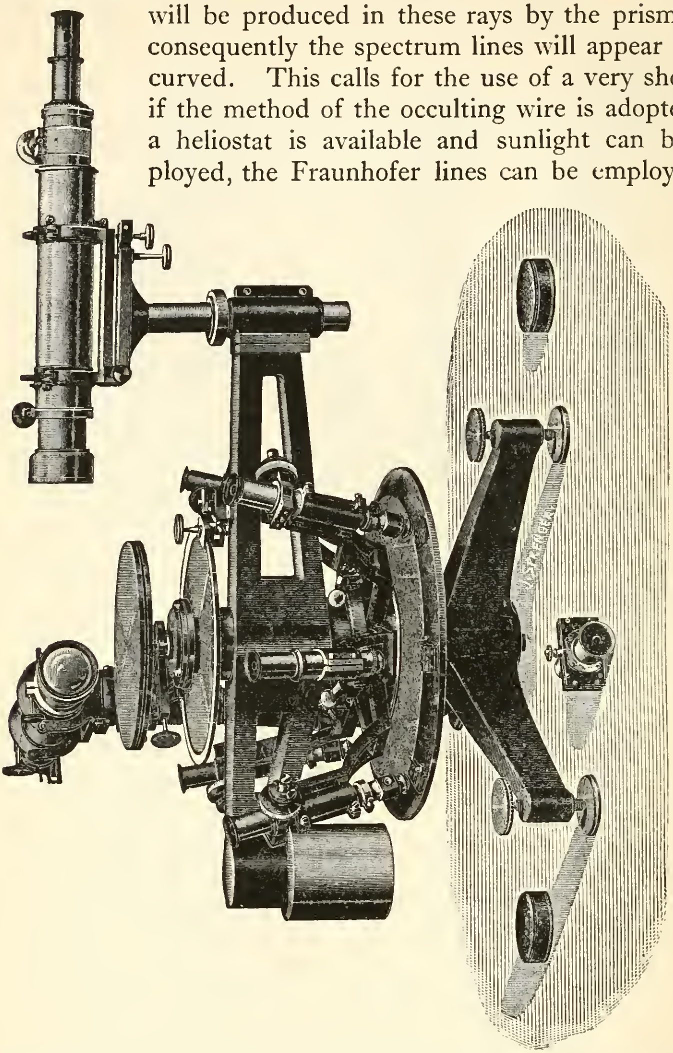

Fig. 79.—Spectrometer by Société Génevoise

Use of Micrometer Microscopes.—For reasons described in a previous chapter the vernier is hardly satisfactory for taking angular measurements when accuracy is desired. In small instruments verniers are used because of their cheapness and convenience, but where such considerations are not vital the provision of micrometer microscopes, even of a small size, for reading the circle would be highly desirable. Some difficulty may be experienced in taking readings with four micrometers through a complete set of measurements, as one is very often found in an awkward position under the collimator. The provision of right-angled prisms in the microscopes (fig. 78 shows a design due to Messrs. A. Hilger, Ltd.) may, however, render the consistent use of two much more easy and convenient. Some trouble should be taken before beginning a series of readings to ensure that no difficulty can arise from causes of this character. It is, of course, absolutely necessary to keep at least to one pair of micrometers for any definite angular measurement. It should be possible, however, to design an instrument in which the circle moves with the telescope and is divided on its under side. It could then be read by four fixed micrometers low enough to escape the telescope arm and carrying right-angled reflecting prisms to render the line of sight horizontal. A spectrometer with four microscopes is shown in fig. 79. This design is due to the Société Génevoise.

A Useful Adjunct.—Failing the provision of micrometer microscopes or even in addition, a useful addition to a spectrometer can be given by providing a tangent screw micrometer for the purpose of moving the telescope arm through small measured angular intervals.

As was previously mentioned, the uncertainty of reading of a scale and vernier in cheap instruments is often of the order of 1′ or 2′ of arc. This means an uncertainty, in the result for refractive index, of several units in the fourth decimal place. Now for the practical refractometry of optical glass it is not only desirable to know the absolute value of refractive index to the fourth decimal place, but the dispersion should be measured with a corresponding accuracy also. Instead of this, dispersion is found from the difference of two results for refractive index which may themselves be in error by several units in the fourth place. As the significant figures for dispersion only commence in the second or third decimal place, an error of several units per cent in the result for dispersion is to be expected.

A graduated micrometer tangent screw permits of the accurate measurement, to a few seconds of arc, of small differences in angular position of the telescope, and thus the absolute value of one angle of minimum deviation, for one spectrum line, may be obtained by the verniers in the usual way, while the remaining angles are found from the differences as given by the tangent micrometer. The result is to produce a much greater accuracy in the measurement of the differences, and thus in the result for dispersion, although the absolute values are subject to the same errors as before. This arrangement was provided in the old "Steinheil" spectrometers of the flat table variety, but has fallen into disuse in more modern instruments.

ANGLE COMPARATORS

It is frequently possible to employ comparator methods for the measurement of prism angles if and when the form of the prism is so good as to permit the assumption that pyramidal error is absent. The defect known as pyramidal error will be understood by reference to fig. 80. The three refracting edges of the prism shown are not parallel. Although the three base angles A, B, and C must add together as 180°, the same is not true of angles α, β, and γ in planes respectively at right angles to the refracting edges concerned.

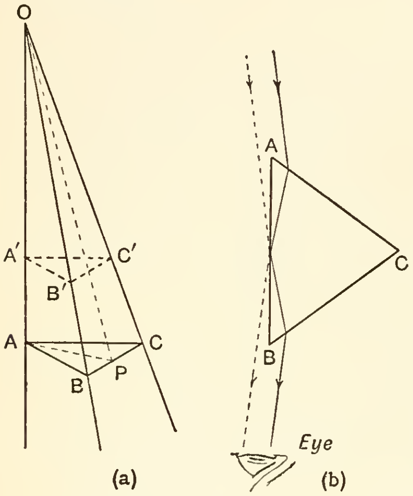

Fig. 80

(*a*) Illustrating Nature of Pyramidal Error.
(*b*) Illustrating Test for Pyramidal Error.

Let the plane ABC be perpendicular to OA. Draw the line AP perpendicular to BC and meeting it in P; also join OP. Then the angle AOP is a measure of the pyramidal error in this particular case.

Guild* shows that, for an isoceles prism (AB = AC), the excess E of the sum of the three angles (α, β, and γ) over 180° may be written

$$E = p^2 \cot ABC,$$

where p is equal to AOP. In the cases of a 60° prism and a 45°, 45°, 90° prism, when p is 10′ E is roughly 1″ and 2″ respectively. When p is as much as 25′ the excess has increased to about 6″ and 11″

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* Guild, *Trans. Opt. Soc.*, Vol. XXII, No. 3, p. 139.

of arc respectively in the two cases. Pyramidal error ought then to be allowed for in some cases, and its amount ought always to be investigated before a prism is used for critical work.

With an isosceles prism one of the most useful tests makes use of the property of such prisms illustrated in fig. 80 (*b*). With a prism of perfect form, parallel rays incident as shown in the figure and suffering external and internal reflection should be parallel on leaving the prism. If, however, pyramidal error is present, the roof edge C will not be parallel to the plane of the face AB. In this case, supposing the edge C to be vertical, the two rays will suffer a relative angular shift in altitude. The test is conveniently applied with the naked eye, which is placed near the edge B so as to receive the two sets of rays simultaneously. In presence of the pyramidal error a distant horizontal window bar would appear doubled, and a separation of 2′ to 3′ should be detected. This would mean a pyramidal error of the order of twice that amount.[*] Guild[†] shows how to perform the test more accurately with the aid of a spectrometer. The measurement of pyramidal error is very simple with the autocollimator.

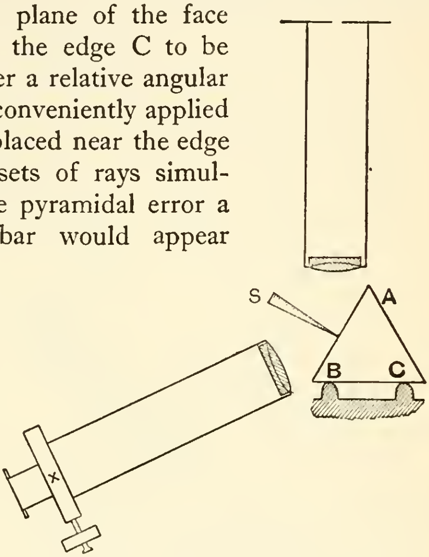

Fig. 81.—Angle Comparator (diagrammatic)

Measurement of Angles (Substitution or Comparator Methods).—If pyramidal error is not present in appreciable amount, some such device as is indicated in fig. 81 may be employed for the comparison of prism angles. One face of the 60° prism rests against a very rigid three-point support. Another face receives light from the collimator and reflects it into the telescope, which is provided with an accurately calibrated micrometer eyepiece. The position of the image is observed. The prism is then turned so that the angle C occupies the position previously held by A. If these angles are not equal the parallel beam will be turned through an angle equal to twice the difference. This angle will be measured by the micrometer and its sign noted. When both B and C have been found in terms of A, the equation

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[*] The test is rather sensitive to defects in the surfaces of the prism, and the precaution should be taken of repeating the observation with the reverse edge near the eye. In the case of serious lack of planeness of the surfaces, or lack of homogeneity in the glass, the two results will not agree. Perfect agreement thus indicates the truth of the prism faces.

[†] Guild, *Trans. Opt. Soc.*, Vol. XXII, No. 3, p. 139.

$$A + B + C = 180^\circ$$

will at once yield the value of A; the values of B and C follow immediately. The precautions described in the description of the prism angle measurement (p. 130) must be remembered. The position of the reflecting face must be controlled by a single stop S.

In a similar way the 45° angles of a right-angled prism could be compared with each other, and if a number of standard prisms of known angles are available, the determination of almost any angle of a frequently occurring type could be carried out by a well equipped laboratory with great convenience and rapidity.

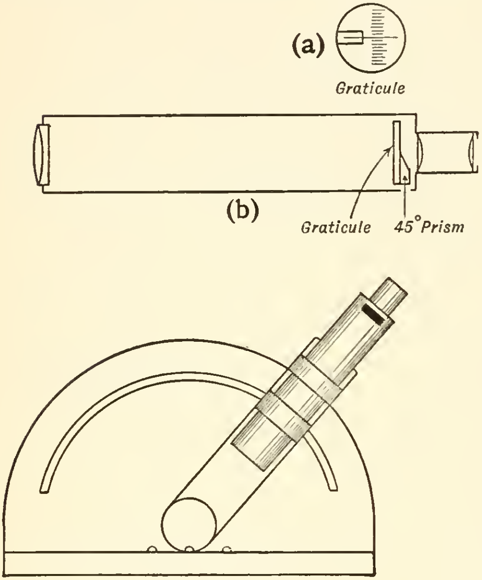

Fig. 82.—Auto-collimator

(*a*) Graticule. (*b*) Telescope. (*c*) Mounting.

Auto-collimator

The principle of the angle comparator is applied very conveniently to ordinary workshop measurements in the auto-collimator (fig. 82), which, to the optician dealing with prisms, is one of the most useful of instruments. It consists of a short telescope with a positive eyepiece which observes a scale in the focal plane. The zero of the scale is preferably in the centre. Opposite the zero is a mark parallel to the scale divisions, and this mark is illuminated by light reflected down the axis of the instrument by a small prism. It is convenient to mount the tube as shown on a firm support, and to arrange so that it can be clamped at any desired angle. Prisms, parallel plates, &c., can be placed on the three-ball support on the base.

The scale in the eyepiece has to be graduated in angular measure. This may be done by measurement of the focal length of the objective and calculating the subtense of one division at that distance, or the scale may be viewed, when mounted in the instrument, by a theodolite looking through the objective; the angular subtense of the whole length of the scale is easily measured in this manner. A convenient length for the objective is about 20 cm., but there is no limit other than that determined by the requirements to be met.

By means of this instrument it is possible to measure the three angles of a 60° prism (to 0·2′ of arc) by a method similar to the comparator method described in the previous section, except that the displacements of the image are read off on the scale instead of being measured by the micrometer.

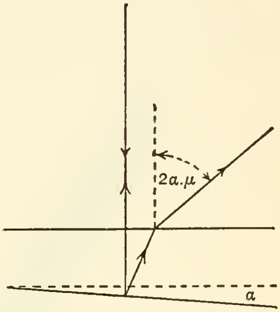

Fig. 83

There are also many approximate measurements which can be made by observing two images reflected by the faces of glass prisms or plates. Fig. 83 shows the case of a plate with two faces inclined at an angle α. Reflection (approximately normal) at the first face gives one reflected beam. Reflection at the back face rotates a ray into a direction making an angle of 2α with the original path. Refraction at the front surface increases the angle to approximately $2\alpha\mu = 3\alpha$, if glass of a refractive index near 1·5 composes the plate.

It will not be found difficult to devise methods for giving a complete test to all the angles of a "90°, 45°, 45°" prism, working on these lines.

Auto-collimating Spectrometers.—The avoidance of the necessity for the awkward collimator in such instruments as described in the previous section is a great convenience, and the principle was applied in the auto-collimating spectrometer of Abbe (fig. 84). In this case a fine slit in the focal plane of the telescope is illuminated by light from a right-angled prism. A reflector has to be used on the table of the instrument. It is very convenient when using liquids, as a hollow prism may be made having angles of 60°, 30°, and 90° in which a reflecting surface backs the perpendicular side. One surface of such a prism can be supported against the face of the vertical prism holder.

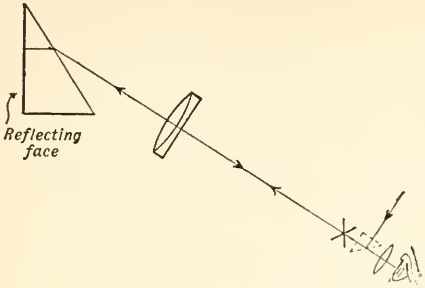

Fig. 84.—Auto-collimating Spectrometer

To find the angle of minimum deviation it is only necessary to move the telescope or prism until the image is found central in the field of view.

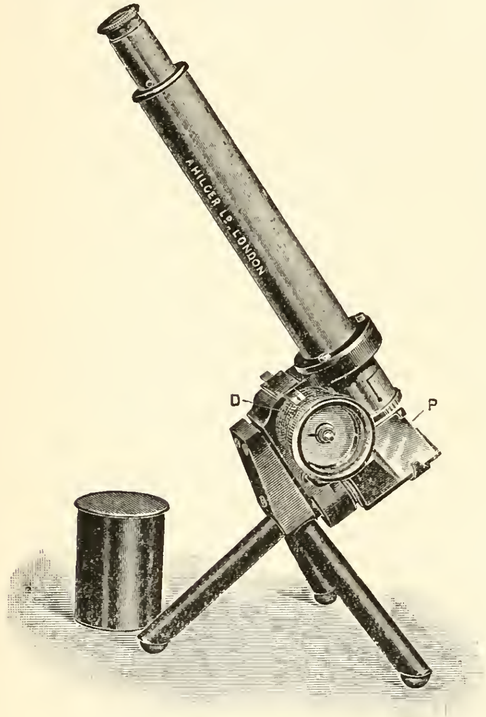

Fig. 85.—Chemical Spectrometer (Hilger)

A convenient form of the instrument—intended, however, for use as a simple spectroscope only—in which the prism is rotated relatively to the telescope by means of a micrometer with divided "drum", is made by Messrs. A. Hilger, Ltd. This replaces the divided circle in the older instruments, and the wave-length of the light is indicated by the drum readings. A mirror is mounted on the tube just outside the slit, and in this way it is easy to observe the spectrum of a flame, for purposes of chemical analysis. The instrument is represented in fig. 85. D is the drum for rotation of the prism P with back reflecting surface.

CHAPTER IX

Refractometers

Refractive Index.—Refractive index is a ratio and has no " dimensions ". Therefore its measurement generally depends on the determination of the value of *two* quantities. Instruments can be designed, however, to fix one variable, when the ratio (refractive index) follows from a simple determination of the other. The accuracy of the result depends much more on the construction and previous calibration of the instrument in these comparative measurements, and their essential difference from absolute measurements must not be forgotten.

Critical-angle Refractometers.—The law of Snellius, usually stated in the form

$$N \sin\alpha = N' \sin\alpha'$$

(where N and N′ are the refractive indices of the media containing the incident and refracted rays, and α and α' are the angles of incidence and refraction), when applied to the case of grazing incidence ($\alpha = 90°$) indicates that under these conditions α' has a real value if N′ is greater than N. The light is then refracted into the medium at the critical angle, for which

$$\sin\alpha' = \frac{N}{N'}.$$

A full discussion of the properties of critical-angle refraction and reflection may be found in textbooks of physics, to which reference should be made, but it may be also recalled here that should the path of the light be reversed and the angle of incidence (now in the denser medium) on the separating surface gradually increased, the reflection becomes complete or " total " at the critical angle, and at this point there is a comparatively *sudden* increase in the intensity of the light reflected. This effect will be referred to when describing certain applications of the Abbe refractometer. Some critical-angle refractometers employ light at grazing incidence; others make use of the above "increase of reflection" effect.

Referring to fig. 86, let N_a be the refractive index of the medium in which light is incident at grazing incidence on a prism; let N_x be the refractive index of the prismatic block with an angle A between its surfaces; and let N_b be the refractive index of the final medium into which light emerges at an angle β.

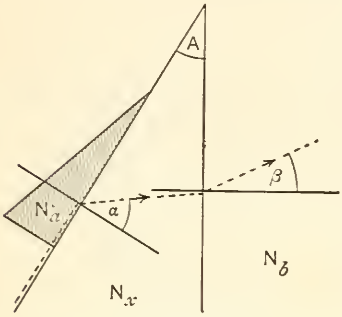

Fig. 86.—Critical-angle Refractometry

Applying the law of refraction to the two surfaces we have

$$\frac{\sin(\alpha - A)}{\sin\beta} = \frac{N_b}{N_x} \ldots\ldots\ldots\ldots (1)$$

and

$$\frac{N_a}{N_x} = \sin\alpha \ \ldots\ldots (2)$$

In order to express N_a (which in refractometry is usually the unknown quantity) in terms of quantities usually known, equation (2) may be transformed thus:

$$\frac{N_a}{N_x} = \sin\alpha = \sin(A + \alpha - A)$$

$$= \sin A \cos\overline{\alpha - A} + \cos A \sin\overline{\alpha - A}$$

$$= \sin A \sqrt{\left(1 - \sin^2\beta\frac{N_b^2}{N_x^2}\right)} + \cos A \ \sin\beta\frac{N_b}{N_x} \text{ (by substituting from equation 1).}$$

In the most frequent cases $N_b = 1$ (for air); hence we obtain

$$N_a = \sin A \sqrt{(N_x^2 - \sin^2\beta)} + \cos A \sin\beta \ldots\ldots (3)$$

In the case of the Pulfrich refractometer the angle A of the prismatic block is frequently 90°. In this case the equation simplifies further to

$$N_a = \sqrt{(N_x^2 - \sin^2\beta)} \ldots\ldots\ldots\ldots (3a)$$

From this equation we could make a table connecting the limiting angle of emergence β with the refractive index of the first medium.

Pulfrich Refractometer.—Figs. 87 and 88 show respectively a cut and diagram of the Pulfrich type refractometer. It is furnished with one or more standard prisms of predetermined refractive index in order to allow for test specimens with different ranges of refractive index. The solid specimens for test must have two plane surfaces very nearly at right angles, and of these the face in contact with the block of the instrument must be a fairly good optical flat. In order to permit of grazing incidence of the light the edge (E, fig. 88) of intersection of the faces of the test specimens must be perfect, and specimens exhibiting any chamfer of the edge have to be recut before testing. A perfect edge is usually secured by making two test prisms at the same time. The first plane polished surfaces made on the two parts are stuck together with wax, and a perpendicular face is then polished simultaneously on each.

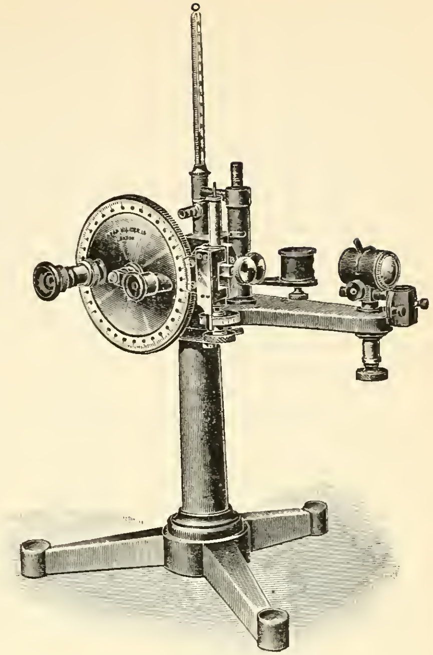

Fig. 87.—Pulfrich Refractometer

In the case of solid specimens for test, a layer of liquid of refractive index higher than the test specimen* must be placed between the prism and block. It is an advantage to make this difference of re-

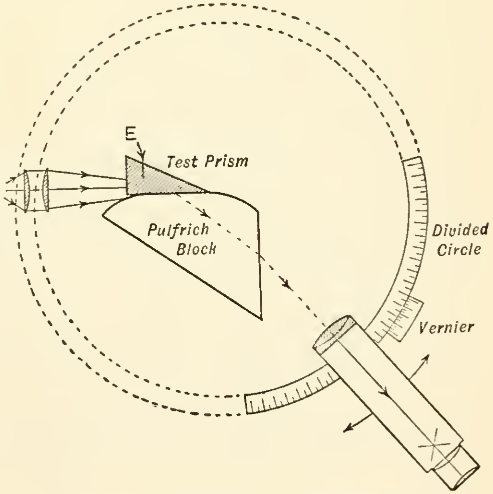

Fig. 88.—Pulfrich Refractometer (diagrammatic)

fractive index small by choosing a suitable liquid having a refractive index as near to that of the specimen as possible.

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* See table, p. 156.

Finally the surfaces of test prism and block have to be made accurately parallel to each other, and this is done by observation of the interference fringes formed by sodium light in the thin film of liquid. Slight pressure on a part of the test prism will usually be sufficient to broaden out the fringes, so that a uniform appearance is presented over the separating surface when the parallelism is sufficiently perfect. It is easier to work with a test piece of rectangular shape if the top face is polished. The fringes should at all events be made parallel to the direction of the incident light. In some cases the fringe system may be seen in the exit pupil of the telescope of the instrument.

If it is desired to use the instrument with liquids, a small cylinder of glass is cemented temporarily to the curved part of the top surface of the block so as in no way to interfere with the grazing incidence of light on the flat central portion. The liquid to be tested is introduced into the cell thus formed.

A convergent beam of light, some rays of which would touch the surface of the block, is thrown into the test prism. The beams emergent from the vertical face of the block may be considered to be sets of parallel beams at various angles, but none of them making an angle with the normal greater than a certain amount (β, fig. 86). The result of this is that when the emergent light is received by a telescope focused for parallel rays, the light, if monochromatic, is found to be confined to one side of a sharp line in the field, which marks the limiting angle. The width of the patch of light beyond the line depends on the angle of convergence of the incident beam, and may, by shutting off part of the aperture of the condenser, be made very narrow, so that a line spectrum is seen when the source is such as will yield it. A shutter is often provided in the condenser for this purpose, and the effect serves to distinguish between the true limiting line and a spurious one on the other side of the patch of light.

In the majority of instruments the telescope rotates with a divided circle moving against a vernier. The centre of rotation should be near the mean position of the image (formed by refraction at the vertical face of the block) of the centre of the top face of the block. The " mean position " of the image is for the range of angles of the emergent light received by the telescope. The setting consists, of course, in setting the cross line or pointer of the telescope on the limiting line in the field.

As it is convenient to have the top face of the block always horizontal, a right-angled prism is usually mounted in front of the object glass of the telescope in order to deflect the light into a horizontal direction. The telescope axis may then be placed at right angles to the divided circle and near its circumference, as shown in fig. 87. As the instrument is not intended for absolute measurements the circle need only be divided through 90°. Any errors of eccentricity, &c., can then be allowed for in special calibration curves.

An auto-collimating device is used for checking the zero setting when the prolongation of the optical axis of the telescope, bent by the right-angled prism, is normal to the vertical face of the block. In one case the image of a small line in the focal plane, illuminated by an auxiliary lamp, has to be brought between two parallel lines engraved on the graticule; the circle should then read 0° 0′, but, if not, the zero correction can be found.

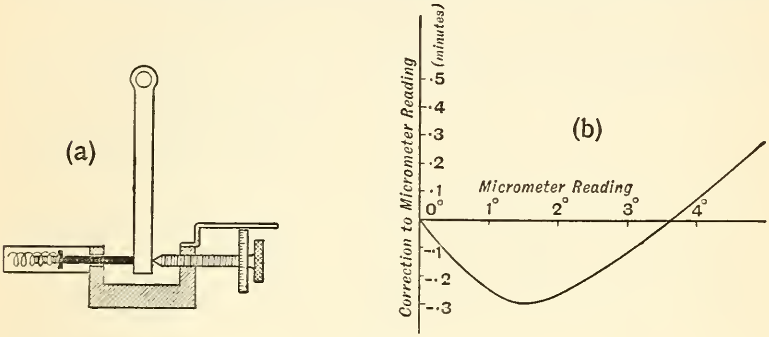

Fig. 89

(*a*) Micrometer Tangent Screw. (*b*) Typical Form of Correction Curve for Micrometer Tangent Screw in a Refractometer.

The telescope and circle may be moved through a few degrees of arc, for dispersion measurements, by means of a micrometer tangent screw, with the help of which "difference angles" may be read accurately to 0·1′ of arc. The screw is usually graduated for a linear movement and gives tolerably correct *angular* readings for a small range somewhere near the centre of its run, but the discrepancy rapidly increases with larger angles. The diagrams (fig. 89 (*a*) and (*b*)) show a micrometer tangent screw and the type of error curve for such a case. Guild* suggests that the bearing plane of the lever arm should be made to pass through the centre of rotation, and that residual corrections might be given in the instrument by shaping the edge of the scale against which readings are made on the micrometer drum head (fig. 90).

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* Loc. cit.

In manufacturing a number of the instruments the blocks will be made from the same melting of glass, and the optical constants of the glass are given by the makers. Tables are then calculated from these constants, from which the values of refractive index corresponding to given angles of emergence for the limiting ray can be obtained for wave-lengths corresponding to the " C ", " D ", " F ", and " G′ " lines of the spectrum. A specimen section of such a table is appended (p. 146). When the instruments are completed, the corrections to each one have to be determined by an examination of prisms of known refractive indices. The results are generally given by means of correcting curves for each wave-length; the correction is to be applied to the refractive indices as obtained from the tables.

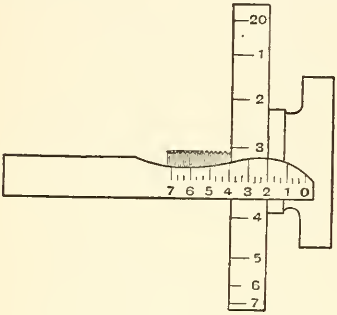

Fig. 90

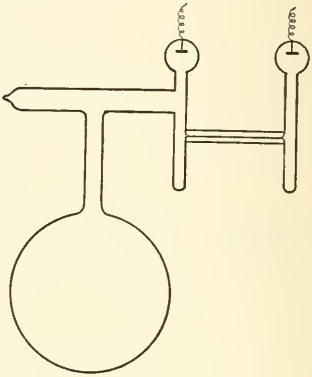

Fig. 91.—Hydrogen Tube

Source of Light.—It is usual to measure refractive indices for the three bright lines usually known as C (red), F (blue-green), and G′ (violet) of the hydrogen spectrum and also for the yellow-orange D line of sodium. The discharge tube is preferably of the " H " form with large reservoir (fig. 91). In a tube of small capacity the hydrogen is liable soon to be occluded by the electrodes, and the spectrum becomes correspondingly feeble.

The " sodium " light is derived from a flame. A very convenient arrangement is a small piece of " asbestos paper " impregnated with common salt and held just in the flame by a " Bulldog " paper clip.

The hydrogen tube is usually clamped immediately behind the condenser, and the sodium light can be brought into the test prism by a right-angled prism so mounted on the instrument that it can be turned into position at will.

The Setting.—The setting is more difficult than is the case in the use of the spectrometer, as the adjustment has to be made on the edge of a band. The type of cross wires or pointer varies between different instruments. In the Zeiss pattern a simple pair of cross wires is given.

As before, it must be remembered that, in making the actual observations, *consistency* rather than absolute accuracy is to be striven after. The conditions used in preparing the calibration curves ought in reality to be identical with those used in making measurements, although this is hardly possible in every case. Differences in the estimated position of the edge of the band are possible with different intensities of the spectrum owing to varying *irradiation*, a visual effect in which the image of a bright object appears a trifle larger than it would with a lessened intensity, owing possibly to a slight spreading of the light to retinal elements contiguous to the true image. Chromatic parallax may be troublesome in the settings, but by using a discharge of fair strength in the hydrogen tube a background of the secondary spectrum of hydrogen will often be sufficiently bright to reveal the cross wires fairly well. In the blue regions of the spectrum the scattered ordinary light may be too strong, but this can be controlled when setting on blue or violet lines by the use of a blue-violet filter between the condenser and the prism.

Special Arrangements for Continuous Work.—In circumstances where a great deal of work is necessary, it will be found an advantage to have a rheostat in the primary of the induction coil by means of which the strength of the discharge can be controlled at will; the G′ line can then be momentarily brightened while a setting is made. It is advisable to work with spectrum lines of approximately equal brightness. At the National Physical Laboratory it was found convenient to produce the sodium spectrum from the vacuum tube by a strong discharge at a pressure of 3 to 5 mm. of hydrogen; under these conditions the glass of the capillary is attacked and decomposed by the discharge, which then glows brilliantly with the sodium light. This method, however, necessitates the use of a rotary oil pump for restoring the vacuum and also a cylinder of hydrogen for an occasional increase of pressure.

Pulfrich Refractometer.—A specimen of the table (part) supplied with an instrument will be found overleaf.

TABLE FOR PRISM NO. ——

N_D = 1·62165. C—D = 0·00488. D—F = 0·01225. F—G′ = 0·02268.

Angle for D.	N_D.	Difference for 1 Min.	Corrections for C, Subtract.	Corrections for F, Add.	Corrections for G′, Add.
32° 0	1·53262	8·5	516	1296	2398
10	177	8·6	6	6	9
20	091	8·6	7	7	2400
30	005	8·6	7	8	2
40	1·52919	8·7	8	8	3
50	832	8·7	8	9	4
33° 0	1·52745		518	1300	2406

METHOD OF ARRANGING THE CALCULATION

Refractive Index.	Dispersion (Micrometer Readings).	Mean Difference.
Vernier reading for D light, 32° 23′·5. From Table N_d = 1·53061.	C 1° 45′·0 1° 17′·6 0° 58′·3 D 2° 9′·5 1° 42′·2 1° 22′·9 F 3° 14′·5 2° 47′·2 2° 27′·8 G 4° 12′·2 3° 44′·9 3° 25′·0	C − D = 0° 24′·6 D − F = 1° 5′·0 D − G′ = 2° 2′·5

	C	F	G′
	32° 23′·5	32° 23′·5	32° 23′·5
	− 0° 24′·6	+ 1° 5′·0	+ 2° 2′·5
	β_C = 31° 58′·9	β_F = 33° 28′·5	$\beta_{G'}$ = 34° 26′·0
From tables:	1·53271	1·52497	1·51988
Appropriate correction:	− 516	+ 1302	+ 2418
	N_C = 1·52755	N_F = 1·53799	$N_{G'}$ = 1·54406

The corrections from the special curves may now have to be added

Accuracy of Results.—The table given above is an extract from one actually given. It will be seen that in this case an error of 1′ of arc in reading the angle on the divided circle would cause an error in the final result of nearly 1 in the fourth decimal place. To calculate the error in the general case the equation (3) may be differentiated. We had

$$N_a = \cos A \sin\beta + \sin A \sqrt{(N_x^2 - \sin^2\beta)}.$$

Hence $$\frac{\partial N_a}{\partial\beta} = \frac{\cos\beta}{\sqrt{(N_x^2 - \sin^2\beta)}}\left\{\cos A\sqrt{(N_x^2 - \sin^2\beta)} - \sin A \sin\beta\right\} \quad (4)$$

This may be reduced by the following steps:

From (3):

$$\sqrt{(N_x^2 - \sin^2\beta)} = \frac{N_a - \cos A \sin\beta}{\sin A}, \ldots\ldots\ldots (5)$$

and, from (2),

$$\cos\alpha = \frac{\sqrt{(N_x^2 - N_a^2)}}{N_x}, \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

which by substituting in (1), expanded, gives

$$\sqrt{(N_x^2 - N_a^2)} = \frac{\sin\beta - N_a \cos A}{\sin A}. \ldots\ldots\ldots (7)$$

On substituting from equation (5) for the square root term in the bracket on the right of equation (4), the part within the bracket reduces to the right-hand member of (7), whence

$$\frac{\partial N_a}{\partial\beta} = \frac{-\cos\beta}{\sqrt{(N_x^2 - \sin^2\beta)}}\sqrt{(N_x^2 - N_a^2)}. \ldots\ldots\ldots (8)$$

Typical examples of the variable will have to be computed to evalute the differential coefficient for a particular case.

To assess the effect, on the final result, of inaccuracy in the prism angle it suffices to differentiate N_a with regard to A. From (3) we obtain,

$$\frac{\partial N_a}{\partial A} = \cos A \sqrt{(N_x^2 - \sin^2\beta)} - \sin A \sin\beta$$

$$= -\sqrt{(N_x^2 - N_a^2)}. \quad \text{(Compare the step from 4 to 8.)} \quad (9)$$

It may be calculated that, for a refractive index of the block = 1·7, an error of 5″ of arc in the estimated angle of the block will cause the determined refractive index to be in error by about one unit in the fifth decimal place. It may also be deduced that there would be little loss of general accuracy of the instrument if the angle A of the prism were made 60°, and the important advantages of the possibility of determining its constants by spectrometer methods, and of testing the prism by interferometer methods for optical imperfections, can then be realized. An instrument embodying several of the above recommendations has been made up by Messrs. Bellingham and Stanley. Points of interest include the provision of a 60° prism, two verniers to the circle, and the arrangement of the tangent screw and bearing plane in the manner previously indicated. To quote the published description: " The micrometer screw for dispersion measurements is fitted with a nut containing an internal spring which takes up any back lash which may be introduced by wear. The lever is maintained in contact with the point of the micrometer screw by a constant-tension C spring." With the C spring the pressure between screw and lever is always constant.

Much of the foregoing discussion has been based on the paper by Guild to which previous reference was made.

Use with Liquids.—The method of use of the instrument with liquids is almost precisely the same as with solids. The special cell and its mounting were referred to on p. 142. The variation of refractive index with temperature is, however, comparatively great for liquids, and a few degrees change in temperature may cause a change in the third decimal place of refractive index.

Thus it is important in a measurement on a liquid to control the temperature of both liquid and block so that the results may be in accordance with standard conditions. To this end a water circulation system is provided in all instruments. The prism is almost entirely surrounded by the water jacket, and a closed cylindrical tube of metal through which the water circulates can be lowered into the cylinder containing the test liquid. It is safe to assume that the liquid between the bottom of the metal cylinder and the top of the prism is at the temperature of the water. This temperature is indicated by a thermometer with its bulb in the cylindrical tube. The circulating water should be kept at a temperature as uniform as possible, and the use of a thermostat is advisable in particular cases for accurate work.

Temperature Corrections for the Prisms.—The refractive indices for a prism block at some standard temperature, *t* say, are usually supplied by the makers, and the temperative coefficient K of the glass should also be given. Then, assuming a simple linear relation, we write

$$N_{t_1} = N_{t_2} + K(t_2 - t_1),$$

where t_1 and t_2 are two temperatures not widely different. We had (p. 140),

$$N_a = \sin A\sqrt{(N_x^2 - \sin^2\beta)} + \cos A \sin\beta.$$

With a given invariable prism angle the value of N_a found by the formula for a given value of β will vary with N_x. By differentiating we obtain,

$$dN_a = \frac{N_x \sin A}{\sqrt{(N_x^2 - \sin^2\beta)}} \cdot dN_x \ldots\ldots\ldots\ldots (10)$$

When A = 90°, the denominator on the right is equal to N_a (from equation 3*a*), and the correction to be applied is therefore:

$$N_a' - N_a = \frac{N_x}{N_a} \cdot (t_2 - t_1) \cdot K.$$

Messrs. Adam Hilger, Ltd., supply, with their instructions for the instrument, a short table showing the corrections to refractive index for temperature in the case of each of the standard wavelengths. The magnitude of the correction with the dense prism is such that the amount to be added, per 1° C. rise above 15° C., to the figure in the fifth place of refractive index is about 0·8 for the sodium line and 1·4 for the G′ hydrogen line.

General Accuracy.—The general accuracy of results obtained with the Pulfrich refractometers is of the order represented by one or two units of the fourth decimal place of refractive index for the absolute value, while the results for dispersion will be accurate within one or two units of the fifth decimal place. These figures apply to refractometry of glass specimens. Greater errors are to be expected with liquids unless elaborate precautions are taken.

The Abbe Refractometer.—This instrument is based upon the same critical angle principle as is used in the Pulfrich refractometer, but the design is intended to make the instrument suitable for quick determinations of the refractive indices (for one mean wavelength) of small quantities of liquid. The range is from about 1·3 to 1·7, and the scale is graduated to read directly in terms of refractive index, the accuracy obtainable being one or two units in the fourth decimal place.

Modifications of the instrument are employed for quick measurements on glass prisms, or on specimens with only one polished face. Dispersion cannot be measured with the instrument, although it can in special cases be roughly assessed.

Fig. 92 is a section of the instrument. As before, a telescope with cross wires in its focal plane is employed to view the dividing line. A standard prism is employed and its refracting angle is 60°. The test substance is in contact with the face remote from the telescope as before, but in the case of liquids the method of illumination or of leading the light from the liquid to the main prism at grazing incidence is changed. Solid test prisms can, however, be used in exactly the same way as with the Pulfrich type of instrument (fig. 93 (*b*)). In this case the telescope and prisms are sometimes rotated about the axis, until the proper surface of the main prism is horizontal. The auxiliary prism is swung clear, and light from a window or sodium flame enters the test prism at grazing incidence. A test plate of glass with polished surfaces at right angles and of predetermined refractive index for sodium light is used to check the indications of the scale.

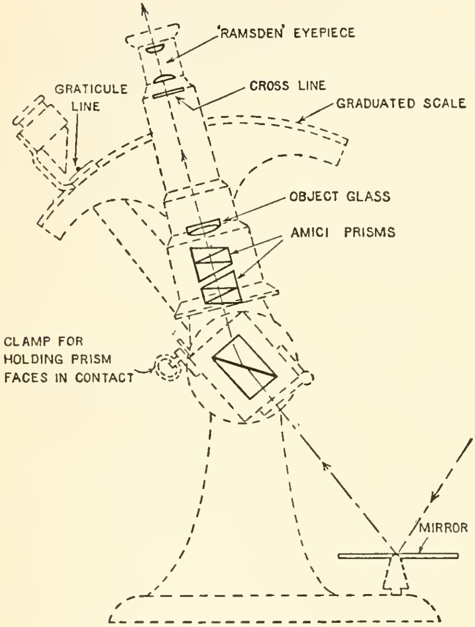

Fig. 92.—Optical System of Abbe Refractometer

When a liquid is tested, a few drops are placed on the surface of the main prism, which is of dense flint glass. The auxiliary prism, which is or should be also of the same glass and has its hypotenuse face ground matt, is then swung round so that the two faces are nearly in contact as is shown in fig. 93 (*c*). A thin film of the liquid is now spread between them. Light from a broad source entering the auxiliary prism at a suitable angle is now scattered into the liquid film, some of it at " grazing emergence ", and, further, enters the main prism at grazing incidence. The result is that the illumination in the focal plane of the telescope is, as previously described, confined to one side of a particular line sharp for each wave-length of monochromatic light. If white light is employed a diffusely coloured border is seen. With sodium light it is easy to set the cross wires on the sharp border line, but as the instrument is intended for quick measurements where a sodium flame may not be always available, two *Amici prisms* are provided to achromatize the border line seen with white light.

The action of an Amici prism will be understood by reference to fig. 94, in which it will be seen that dispersion occurs although no deviation is produced in the mean central ray. The prisms A and C are of crown and B is of flint glass. The prisms actually used in the refractometer have much smaller refracting angles and are therefore comparatively thin.

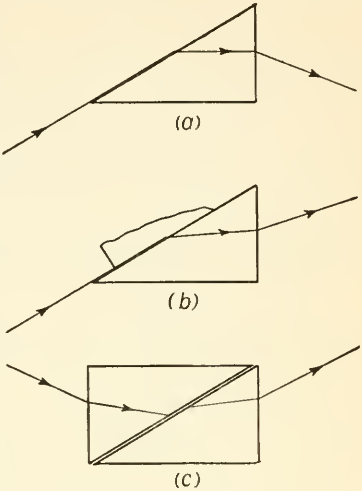

Fig. 93.—Grazing Incidence Refraction in Three Cases

The two prisms which are provided in the Abbe refractometer can be made to rotate in an opposite sense about the optic axis of the instrument by means of a small bevelled cog-wheel. When they are oppositely placed the dispersion produced by one is completely annulled by the other, but it will be seen that any dispersion up to double the amount produced by a single prism can be produced in either direction. With the help of these prisms the border line may be achromatized for two colours, and when this is done it appears sharp enough to " set upon ", although a certain amount of secondary spectrum is usually present. For work with particular substances varying only slightly in composition it may be possible to treat

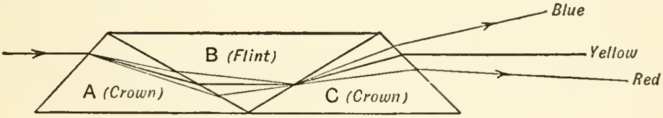

Fig. 94.—Amici Prism (the dispersion is exaggerated)

the rotation of the prisms required to achromatize the border line as a rough arbitrary measure of dispersion, but it is useless to seek to compare the dispersions of, say, liquids and solids in this way.

The mechanical construction is very simple. The prisms can be moved by hand (relatively to the telescope) to make the setting. Extending from the prism casing is an arm, to the end of which is attached the pointer moving over the scale. In most cases the pointer consists of a small glass " cursor " with a fine line engraved upon it. In order to correct any zero error the arm may be released from the prisms by loosening the locking nut. As was previously mentioned, the whole system of prisms and telescope can be turned together on the axis when desired.

With an instrument of this type the accuracy of the results is entirely dependent on the construction. It finds its main use for commercial purposes—in the testing of the purity of oils, &c.

Effects of Temperature.—As with the Pulfrich instrument, the control of temperature is of great importance, and the prisms are surrounded by water jackets.

The effect of the variation of temperature on the indications may be calculated with the aid of equation (10) of the present chapter if the optical constants and temperature coefficient of the prism are known. For most purposes, however, the temperature correction will not be required, as it will be small in comparison with the remaining errors unless the temperature is above about 20° C. Prolonged heating of the prisms is not advisable, as in some instruments they are liable in these circumstances to sink in their mounts, owing to the softening of the cement, or to the loosening effects of expansion.

Novel Features in Design.—Bellingham and Stanley have also introduced certain novel features into the design of the Abbe refractometer. The auxiliary prism is completely detachable, so that glass specimens, &c., can be examined without swinging the contact face into a horizontal direction. Further, the auxiliary prism is of dense glass. This is not the case in some other types of the instrument, and the prism thus fails to lead light into the layer of liquid sufficiently nearly at grazing incidence, when the liquid is of comparatively high refractive index. Bellingham and Stanley also provide an ingenious arrangement for preventing looseness in the racks and pinions for rotating the Amici prisms.

Abbe Refractometer for Glass Specimens.—One type of the instrument may be employed for testing glass specimens with only one polished surface. Thus it is not possible to employ the grazing incidence method. In this case the face AC of the block prism (fig. 95) is ground matt so that it diffuses light at a range of angles on to the face AB. As before, a film of liquid of refractive index higher than the specimen S is interposed between the glass surfaces, but the action on the angles of reflection is identical, from the optical point of view, with that which would result if the film were absent and the two glass faces in optical contact. When the ray is incident on the face AB at an angle which is gradually increased through the value $\sin^{-1}\frac{N_s}{N_p}$, where N_s and N_p are refractive indices of specimen and block respectively, there is at this point a comparatively sudden increase in the intensity of reflected light. The result is that, when the light is received by a telescope as before, the field of view is divided sharply into two portions between which there is a sudden step in brightness. The position of the dividing line locates the critical angle. The grazing incidence method may be employed alternatively if a suitable test prism is available.

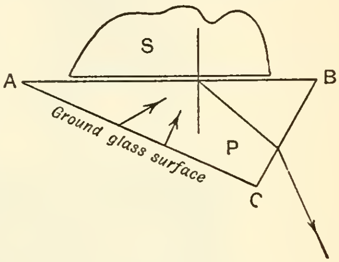

Fig. 95

The use of such an instrument being limited to glass specimens, the water jacket for the prisms may be dispensed with.

Critical-angle Refractometers of Limited Range.—Optical methods of control are becoming of increasing importance in industry. Very frequently the refractive index of a product gives a sensitive indication of its state of purity. For an examination of one particular substance it is clearly not necessary to employ an instrument having a large range of readings. For many purposes it suffices, in a type of refractometer similar to the Abbe, to employ a telescope fixed relatively to the standard prism. The dividing line then moves in the field of view according to the variation of refractive index of the test substance, and its position is indicated by a graticule placed in the focal plane of the eyepiece. This graticule may be graduated directly in terms of refractive index.

In the Zeiss *Butyro* refractometer the scale (an arbitrary one in this case) is divided into 100 parts, and $\frac{1}{10}$ of a division can be read by estimation, corresponding roughly to one unit in the fourth decimal place of refractive index. The position of the graticule is adjustable by means of a small key. Water jacketing is provided for the prisms, as temperature control is of great importance in this case. For the limited range employed the Amici prisms are dispensed with, and the dispersion produced when the light leaves the film of test liquid is compensated by that produced at the last surface of the standard prism block.

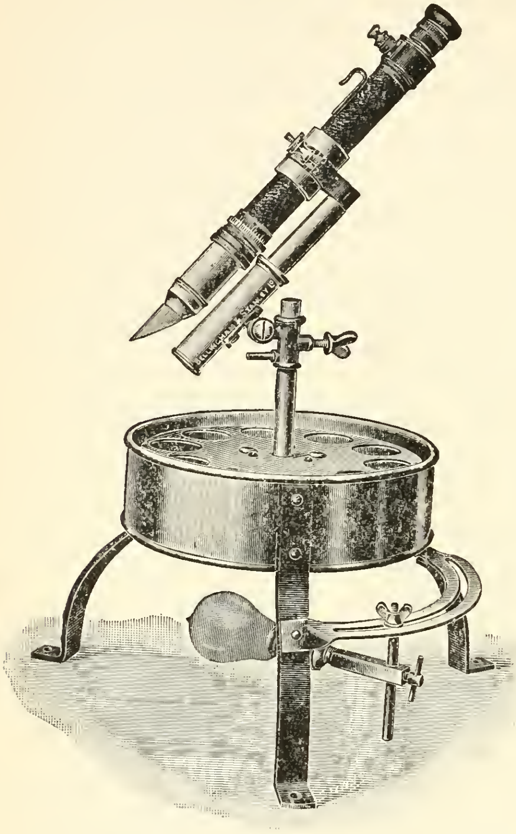

Fig. 96.—Dipping Refractometer

Dipping Refractometers (fig. 96 shows a pattern due to Messrs. Bellingham & Stanley) are being found of increasing use in cases where more than a small quantity of liquid is available, so that the test prism, suitably protected by jacketing, may be dipped bodily into the liquid for test. Light is received at grazing incidence by the test prism and usually comes through the transparent base of the containing vessel. The temperature of the liquid has, of course, to be controlled. The range of a typical instrument is from 1·325 to 1·367, which suffices for a considerable number of typical beverages. Additional prisms can be procured for use of the instruments over other ranges. Some instruments of this type are fitted with graticules only, while others possess an auxiliary micrometer of a suitably modified form.

Failure to obtain a Sharp Border Line.—Not infrequently it is found that certain oils, &c., fail to produce a sharp border line with a critical-angle refractometer. This is usually ascribed to the non-homogeneity of the product, in which several substances of varying refractive index may be fairly well mixed, although not sufficiently mixed in the molecular sense to give a really optically homogeneous liquid. In such a case it may be possible to secure more thorough mutual solution or to separate some of the products.

Crystal Refractometers.—The refractometer is of considerable importance in mineralogy, as there is no more rapid and successful means of identifying crystals with one plane face of sufficiently good optical quality. Several types of such instruments are available which depend on the critical-angle principle, but the standard block is usually hemispherical and no longer prismatic (fig. 97). In this case the mean ray derived from the centre of the upper flat face suffers no deviation, and the vernier attached to the telescope, which rotates about an axis passing through this same point, may be made to read the critical angle directly on the circle of the instrument.

The refraction of the light at the spherical surface of the block causes a collective action on the rays. The "objective" of the telescope needs therefore merely to be computed so that the whole system is spherically and chromatically corrected. Either the grazing incidence or critical-angle reflection methods may be employed, the instrument being intended for use with monochromatic light.

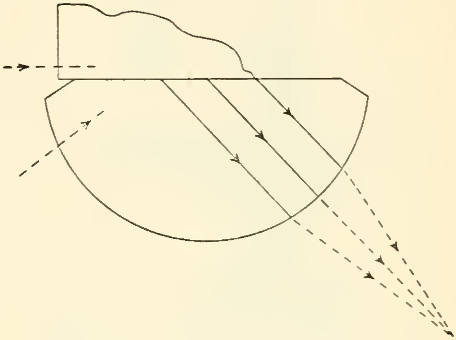

Fig. 97.—Action of Hemispherical Block

Various means are adopted in different instruments for bringing the line of sight through the telescope into a convenient position. In the large Zeiss instrument (fig. 98) two prisms were provided which bring this line of sight into the axis of the vertical circle. For convenience in showing the section, the reflecting prisms in the "telescope" system have been rotated so as to be above the hemispherical prism. In this instrument the hemisphere must fulfil the following conditions: (1) the axis of the hemisphere must be at right angles to that of the vertical circle; (2) both axes must intersect; (3) the centre of the hemisphere must be coincident with the axis of the vertical circle.

It is a great convenience to be able to rotate the standard prism while the crystal under test is mounted in position, since the refracture index of crystals may vary with the direction of transmission of the light.

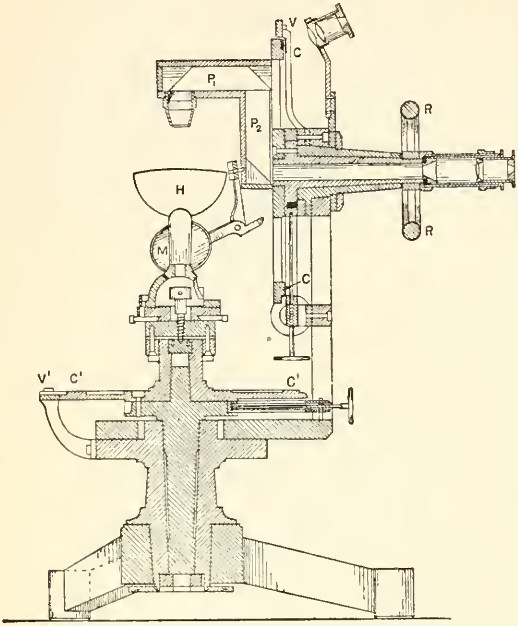

Fig. 98.—Cross-section of the Crystal Refractometer

C, Vertical circle. C^1, Horizontal circle. H, Standard block. P_1, P_2, Reflecting-prisms. M, Illuminating mirror. R, Adjusting ring. V and V^1 Verniers.

A much smaller instrument, almost of the pocket type, has now been brought out by Zeiss in which refractive indices to 1·84 are read off by a scale in the focal plane of the eyepiece (by estimation to the third place of decimals). The vertical circle is thus dispensed with, and there is no objective.

Liquids of High Refractive Index.—The crystal refractometer will be of no use for the measurement of high refractive indices unless liquids of higher refractive indices are available. The following table gives a number of suitable liquids.

Substance.	Refractive Index for Sodium Light.
Aniline	1·56
Quinoline	1·62
Monobromnaphthaline	1·66
Mercuric iodide dissolved in an aqueous solution of potassium iodide (saturated)	1·71
Methylene iodide	1·74

For the examination of minerals with higher refractive indices than can be dealt with by the crystal refractometer, a number of approximate methods have been developed in which liquids of still higher refractive indices are employed.*

Differential Refractometers.—There are a number of methods which permit of an accurate comparison of the refractive indices of substances, although they are not always suitable for "absolute" determination. The most sensitive of such methods is exemplified by the gas interference refractometer, but there are a number of differential refractometers for liquids which merit attention.

Fig. 99 (*a*) illustrates a differential critical angle method due to Hallwachs.† A double cell with walls of plane parallel glass and with a separating wall also of plane parallel glass contains in its two halves the liquids of which the refractive index is to be compared. Light is caused to enter at grazing incidence on the separating wall. (This must be on the side in which the liquid has the lower refractive index, and the refractive index of the glass must be higher than that of the liquid. Let μ_1 and μ_2 be the refractive indices of the liquids in compartments 1 and 2 (fig. 99 (*a*)), and let i and i' be the angles of refraction of the ray in the second liquid and air respectively; then the plane parallel walls may be treated as optically non-existent.

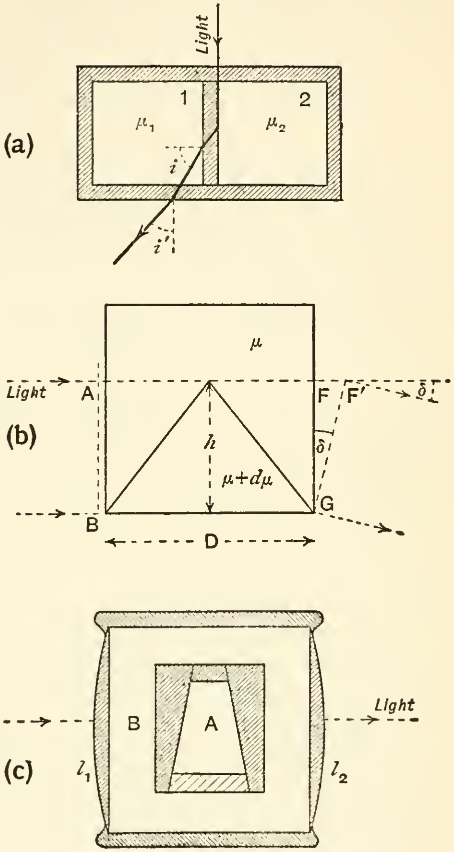

Fig. 99.—Differential Refractometers

(*a*) Hallwachs. (*b*) Anderson. (*c*) Féry

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* For a description of these see *Methods of Photographic Research*, Pub. Carnegie Inst., Washington, No. 158, or Johannsen's work on *Petrographic methods.*

† *Trans. Opt. Soc.*, Vol. XXII, No. 3, p. 156.

We have $$\mu_1 \sin i = \mu_2,$$
and $$\mu_1 \cos i = \sin i'.$$

Squaring and adding these equations,

$$\mu_1^2 = \mu_2^2 + \sin^2 i';$$

hence $$\mu_1 - \mu_2 = \frac{\sin^2 i'}{\mu_1 + \mu_2}.$$

A critical-angle border line may then be found by using a telescope to receive the light passing through the cell. The apparatus employed to register the angular displacement will obviously depend on the magnitude of the effect. Anderson* states that for liquids having refractive indices near that of water, a difference of refractive index of 1 in the fifth decimal would lead to a displacement of the border line through about $\frac{1}{3}$ of a degree. Ample accuracy for most purposes would be secured by mounting the prism on the table of a spectrometer. It should be noted that temperature precautions are of the greatest moment, as a variation of 1° C. would be likely to cause a change in the refractive index of a liquid of the order of one or more units in the fourth decimal place. If the partition wall is slightly prismatic the error may be eliminated by turning the cell through 180°, measuring the double deviation and taking half this as the true value of i'.

As constructed by Bellingham and Stanley, the cell is of plane parallel glass with the touching surfaces held by clamps in optical contact, no cement being employed.

Anderson goes on to describe a differential refractometer of the form illustrated in fig. 99 (*b*), and gives a formula, derived on the basis of ray theory, which can be employed to measure comparatively large differences of refractive index, but a simple formula which will be sufficiently accurate for small differences can be obtained from the consideration of optical paths. The cell walls may be supposed as before to be constructed of plane parallel glass, and their effects need not then be further considered. The larger part of the cell is filled with liquid of refractive index μ, and the inner prism-shaped part with liquid of refractive index $\mu + d\mu$.

Imagine, however, that the inner prism is first filled with liquid of refractive index equal to that of the outer part. Then a plane wave transmitted through the whole prism will remain parallel to its first direction. AB is the wave-front before incidence and FG the wave-front on emergence. The optical path $\mu.AF = \mu.BG$. (Let $AF = BG = D$, say.)

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* *Trans. Opt. Soc.*, Vol. XXII, No. 3, p. 156.

If now the refractive index of the inner portion is raised the emergent wave-front will make angle δ with its first direction. The optical path between A and F, which we consider to pass only through the tip of the prism, is not affected by the change. Therefore from a simple consideration of Fermat's theorem * we may write:

$$\mu.AF + FF' = \overline{\mu + d\mu}BG,$$

where F′ and G are on the new wave-front, or

$$\mu D + h\delta = \overline{\mu + d\mu}D, \text{ very nearly,}$$

where h is the height of the prism. Hence,

$$\delta = d\mu \, . \frac{D}{h},$$

and this formula is applicable even when the cell is not symmetrical. Clearly when the angle of the interior prism = 90° the formula will become:

$$\delta = 2d\mu.$$

Anderson calculates that the difference in angle produced by a change of refractive index of 1 in the fifth decimal place will produce a deviation of about 4″. As in the previous example, the cell may be placed on the table of a spectrometer, when the angle between the direct and deviated images may be ascertained. "Temperature" precautions are again of great importance. It is, however, simpler to employ a collimator with a straight cross line in its focus. This object may be illuminated by monochromatic light and observed through the prism by a telescope with a graticule, calibrated in angular terms, or a micrometer thread in its eyepiece.

Féry Refractometer.†—No notice of liquid refractometers would be complete without a reference to that of Professor Féry, which is fairly widely used where sufficient quantities of the liquids to be tested are available. It is capable of giving results accurate to about 1 in the fourth decimal place of refractive index, and, as constructed by Messrs. Adam Hilger, Ltd., its range is approximately from 1·33 to 1·6726. It consists essentially of a collimator and telescope between which is placed the cell illustrated in fig. 99 (*c*). The outer walls are low-power plano-convex lenses l_1 and l_2, and the space B is filled with water. The inner prismatic cell of glass contains the liquid to be tested in the space A, and this inner component forms a prism of which the power depends on the refractive index of the liquid. The power will evidently be zero when the refractive indices of glass and liquid are equal. Under the latter conditions a central ray will be transmitted without deviation, but should the power of the inner prism become finite it will still be possible to annul the deviation by moving the cell at right angles to the direction of the light. This will introduce a deviation, caused by the eccentric passage of the ray through the lenses, which is to a first approximation proportional to the displacement of the ray from the axis.

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* By Fermat's theorem the path from A to F′ (the latter point being on the new wave front GF′, after the small first order change of refractive index), measured along the old ray track, differs only by a small quantity of the second order from the true path from A, measured correctly along the new ray tracks, to the new wave front.

† C. Féry, *Comptes rendus*, **113** (1891), 1028; also **118** (1894), 332.

It thus appears that since the deviation produced by the inner prism is proportional to the differences of refractive indices of liquid and glass (see the proof given in connection with Anderson's refractometer), the scale of refractive index for the Féry refractometer may be given by a vernier registering the linear displacement of the compound cell at right angles to the direction of the light.

In the actual instrument the vernier registers to $\frac{1}{300}$ mm., which corresponds to 1 in the fourth decimal of refractive index. The apertures of collimator and telescope object glasses are naturally narrow, and the lenses are of such power that parallel light can be passed through the inner prism. Sodium light is employed.

The glass employed in the construction of all such differential prisms has to be of specially resistant crown glass, and care should be taken not to leave liquids in contact with the surfaces longer than is necessary. Distilled water is one of the most active substances in attacking glass surfaces.

Interferometric Refractometers for Gases.—The refractive indices of gases are usually so small that methods involving measurements of angular refraction do not lend themselves readily to such determinations, and interference methods are of greater value. A number of types of interferometer can be used for such work, but the majority require somewhat delicate manipulation. A type developed by Ramsey, Travers, Lord Rayleigh, Haber, and others lends itself, however, more readily to technical use, and instruments on these lines are now manufactured by a number of firms.* We select for description the laboratory gas interferometer of Messrs. Carl Zeiss; this is illustrated diagrammatically in fig. 100, which shows a plan and elevation. Consider first the elevation (fig. 100 (*b*)), imagining that the whole aperture of the lenses A and B is used. S is a short vertical slit at the principal focus of a collimator lens. " Parallel " light passes to the telescope lens through a number of plane parallel pieces of glass (which do not destroy its parallelism), and finally the telescope lens forms an image of the slit at *s*′. Immediately behind this is a short semi-cylinder of glass C, which has its axis vertical and therefore possesses no dioptric power in a vertical direction. If it were removed, an eye placed behind the small stop as shown, and looking at the back lens, would see that lens uniformly illuminated, because light is converging from all parts of the lens to the aperture. The light coming from the upper portion of the lens has all travelled above the axis of the system, passing through the plane parallel plate H, whereas light coming from the lower portion has travelled through a gas chamber and a compensating plate (P or Q). The division of the field therefore corresponds always to these two parts of the beam. Should the cylinder be placed in position when the whole aperture of the lens B is used, the image of the slit would be seen magnified, but in the horizontal direction only; the appearance is thus of a patch of light limited in the horizontal direction by the size of the magnified image of the slit, and divided by a horizontal line into two parts corresponding to the upper and lower parts of the beam; (a horizontal line on

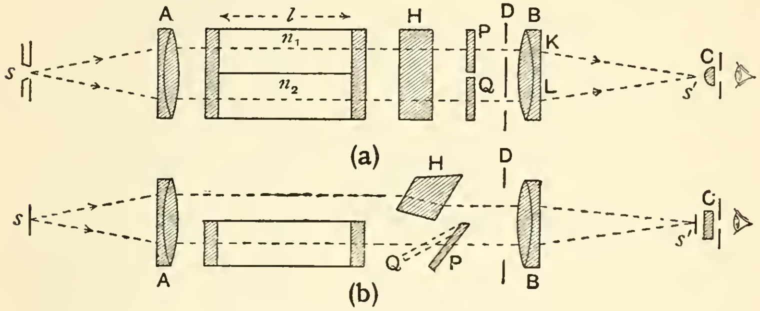

Fig. 100.—The Gas Refractometer

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* Gas interferometers on somewhat similar lines are manufactured in Great Britain by Messrs. A. Hilger, Ltd., who also make other forms of interferometers for use in industry.

the lens B would still be seen by the eye, but a vertical line would not).

The upper horizontal wall of the gas chamber, especially if not quite parallel to the light, would be seen as a dark line in the field. The plate H, however, can be used slightly to displace the upper beam in a downward direction so that the dark edge can be hidden. The effective thickness of H corresponds to the thickness of the glass ends of the gas chamber and the compensating plate P or Q.

The plan (*a*) may now be considered. Notice first that the whole aperture of the lens B is not employed, but is limited by two fairly wide vertical slits in the diaphragm D. The effect of this is to produce a system of vertical interference fringes instead of the simple image at s'. The division of the field is, however, still as before. The gas chamber is in two portions; also P and Q can be given relative rotation about an axis coincident with their upper edges.

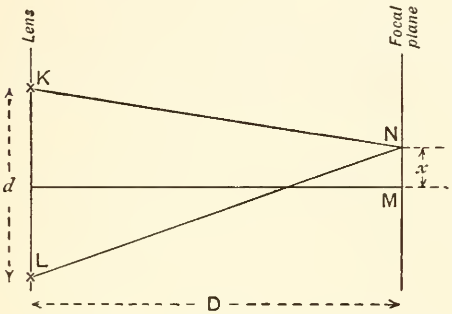

Fig. 101

When white light is used a system of coloured fringes is seen. The "central" fringe only is white and is bordered by black interference bands in which no colour is visible. The dark bands in the outer fringes all show more or less colour, so that the central or white fringe can easily be recognized. This fringe is found at the point in which there is an equality of optical path in the two routes by which the light can arrive at s' from s. The relative optical path length in the two routes may, however, vary by change of the refractive indices in the gas chambers or by slight relative rotations of P and Q. This affects the lower part of the field only. If variation takes place the white fringe in the lower part will be displaced relatively to the white fringe in the upper part where *no* change takes place.

Let l be the common length of the gas chambers, which may be supposed to be filled with gases of refractive index n_1 and n_2 respectively. The difference of path introduced is thus $l(n_1 - n_2)$. Referring to figs. 100 and 101, let K and L represent points on the back surface of the telescope objective where the beams from the two apertures emerge from the lens.

If N (fig. 101) is the position of the white fringe when a difference of path has been introduced in the gas chambers, then LN − KN = $l(n_1 - n_2)$. The difference of optical path is evidently very nearly compensated by this condition, although the presence of the lens would modify the action.

It may be shown without difficulty that $\text{LN} - \text{KN} = \dfrac{d \, . \, x}{\text{D}}$ (very nearly) where $d = \text{KL}$, $x = \text{NM}$, and D is the distance from the lens to the focal plane, provided that d is fairly small in comparison to D.

From a measurement of the displacement of the white fringe and a knowledge of the constants of the system, it would thus be possible to calculate the difference of refractive indices above, but in practice it is found more convenient to use a null method giving a much larger range.

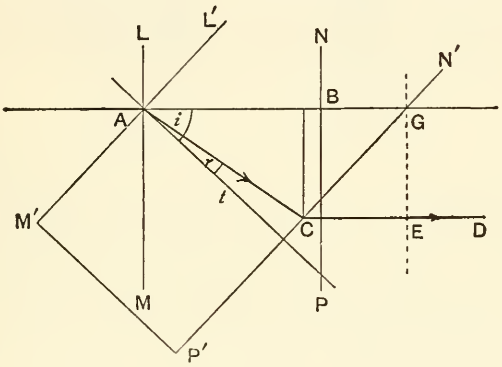

Fig. 102

The relative path of the two parts of the lower beam may be varied by tilting one of the compensating plates P by means of a micrometer screw and a long lever. The reading of the micrometer head thus furnishes an arbitrary scale expressing the optical path difference. If the tilting motion can be calculated in angular measure it is easy to compute the difference of path caused by a given rotation of the compensating plate. The calculation may be given here as it is of frequent use in interferometer methods.

Referring to fig. 102, LM and NP are the walls of the plate in one position, while L′M′ and N′P′ are the walls in the inclined position. The normal path AB inside the plate of thickness t is μt, where μ is the refractive index of the glass. The new direction of the last surface cuts the original ray (AB produced) in G. Draw from G the perpendicular cutting the new path of the ray in the point E. Then in each case we need only consider optical paths up to the line GE.

The optical paths are (1) (before tilt):

$$\mu t + \frac{t}{\cos i} - t.$$

(2) (after tilt):

$$\frac{\mu t}{\cos r} + \frac{t \sin(i - r) \tan i}{\cos r}.$$

Hence the change of optical path introduced by the tilt may be written, after an easy transformation:

$$\mu t(1 - \cos r) - t(1 - \cos i).$$

In the normal position the compensating plates are held at 45° to the incident light and the path may thus be increased or diminished by this means. Messrs. Zeiss recommend that the compensator scale should be calibrated specially by trial for special purposes, mixtures of varying but known difference of refractive index being used in the gas chambers.

The dimensions of the Zeiss laboratory instrument are approximately as follows: length of gas chambers, 100 cm.; cross section of gas chamber, 1 sq. cm.; over-all dimension, 10 cm. wide, 20 cm. high, 200 cm. long.

The refractive index of a gas is greatly dependent on temperature and also on the presence of water vapour. Special drying tubes through which gas must pass before entering the gas chambers are therefore supplied with the instrument. For precise work on gas refractometry it is convenient to work in a " constant temperature room ".

An important application of the gas refractometer is for the detection of fire-damp in coal-mines. For this purpose the design of the instrument is altered so as to make it portable, the alteration being to employ an auto-collimating optical system. Referring to fig. 100 it will be seen that the slit *s* might be placed close to *s*′; the lens B then becomes a collimator. A mirror with a diaphragm containing two vertical slits in front of it is placed in the position A, and the light is then returned almost along its own path, the final images being produced at s' as before. The instrument (fig. 103) is supplied complete with drying tubes and small portable accumulators (containing glass wool to hold the acid) for the 4-volt lamps employed as sources of light both in the instrument and for taking micrometer readings. The case is sometimes made of a shape suitable for strapping to a miner's back. This is one of the most notable examples of the way in which instruments of the greatest delicacy may be adapted for employment under rough conditions of usage.

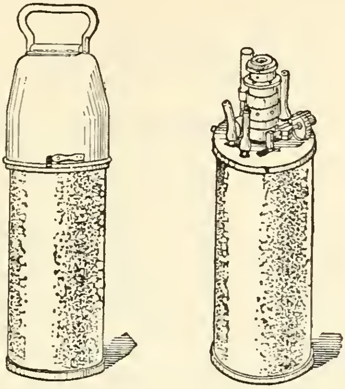

Fig. 103.—The Portable Gas Interferometer, shown with and without Carrying Guard Hood

CHAPTER X

Spherometers and Instruments for Measuring Curvature

Curvature Measurements.—At the present time the great majority of optical systems are constructed with spherical surfaces which are produced with the aid of spherical grinding tools. Consequently in optical work the measurement of curvature is of great importance. Varying methods are employed, depending on the radius of the curve to be ascertained. With comparatively flat surfaces of long radius (over 1 m. or so), it is best to employ the method of measuring the system of interference rings (Newton's rings) formed when the surface is in contact with a good glass " flat ". Details of the measurement will be found in text books of physics, or in the paper to which the footnote* refers. The various types of spherometer are suitable for radii of between 1 m. and, say, 2 cm., the lower limit depending on the diameter of the smallest ring which may be available for use with the instrument. For surfaces with still smaller radii several methods are available, and one special instrument will be described in the present chapter. It may be noted at the outset that when a complete sphere or a portion greater than a hemisphere is available the measurement of the diameter by the aid of micrometer or other calipers is easy and exact. A number of methods have been developed in recent years by means of which surfaces with only a small portion of the sphere remaining may be *compared* with specimens in which a complete diameter can be measured.

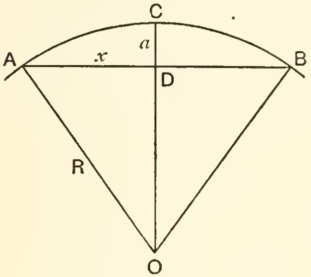

Fig. 104

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* J. Guild, " On Curvature Measurements ", *Trans. Opt. Soc.*, Vol. XXII, No. 3, p. 127.

The Spherometer.—Let ACB (fig. 104) represent the principal section of a spherical surface and AB the trace of a plane, perpendicular both to the principal section and to the radius OC. The spherometer fixes the diameter of the circle AB as a constant of the instrument and measures the sagitta CD. We have

$$AD^2 = CD(2R - CD),$$

where R is the required radius. Let AD $= x$ and CD $= a$; then

$$R = \frac{x^2 + a^2}{2a} \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

In performing the measurement the sagitta CD has to be determined by the aid of a micrometer screw, or by means of a scale and reading microscope. One great difficulty is due to causes which were touched upon at the end of Chapter III, namely, the uncertainty of the point of contact between a plunger and a surface. Successive settings have to be made with an optical flat and the test surface respectively in position on the instrument. Another difficulty, in many cases more serious, is in ascertaining exactly the true diameter of the standard circle AB; for, in instruments which have a metal circle on which the surface rests, the edge of the rim cannot be made indefinitely sharp.

Effect of Errors.—Differentiating the above formula it is found that:

$$\frac{\partial R}{\partial x} = \frac{x}{a}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\frac{\partial R}{\partial a} = \frac{2a(2a) - 2(x^2 + a^2)}{4a^2} = 1 - \frac{R}{a} \ldots\ldots\ldots (3)$$

Now it may be supposed that x can be measured by micrometer microscope or by other means to about one hundredth part of a millimetre. To find the error in R produced by an error of this order in x, we may take a possible case when $R = 10$ cm., $x = 2$ cm., and $a = 0{\cdot}2$ cm., approximately. Then $\frac{dR}{dx} = 10$. This shows that in this case an error of 0·01 mm. in determining x would produce an error of 0·1 mm. in the result for R, an error of one part in 1000. In the same case $\frac{dR}{da} = -49$. It will generally be possible to measure a nominally to within 0·001 mm. An error of this order will produce an error of 0·05 mm. in R or 5 parts in 10,000. It appears, however, that in many cases the determination of x is the chief source of trouble when R is comparatively small (say less than five times the radius of the standard circle). It will be instructive to work out the values of the relative errors in particular cases which may be encountered.

Three-legged Spherometer.—The spherometers of this type found in physical laboratories for teaching purposes are usually little more than toys. The mean radius is best found by measuring, with a micrometer microscope, the mean of the three intervals between the centres of the outer legs and dividing by $\sqrt{3}$ to obtain the mean radius. If, as is usual and as is suggested in fig. 105 (*a*), flats have been worn on the outer feet, the radius of the flats can be measured at the same time. It will be noticed that it is better to measure over the outer legs, as the appearance of the end of the central plunger is generally different from the others and its centre is hard to " set " upon. If h is the radius of the circular worn foot, the effective radius of the standard circle will differ by an amount $\mp h$ from the mean according as a convex or concave surface is being measured. The formula may be written

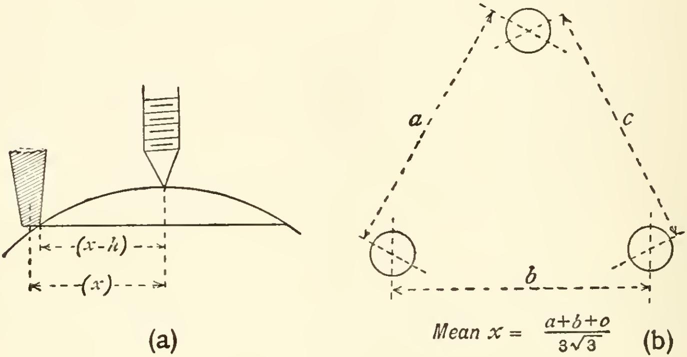

Fig. 105

$$R = \frac{(x \pm h)^2 + a^2}{2a},$$

in which the sign of h is positive for a concave and negative for a convex surface, and x is the standard radius to the centre of the outer feet. See fig. 105 (*b*).

When the sagitta of the curve is measured by a micrometer

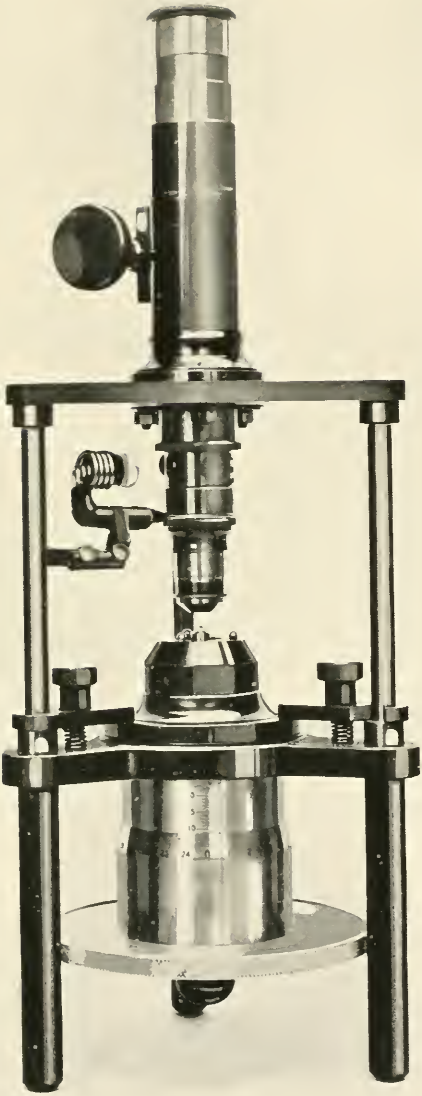

Fig. 106.—Guild-Aldis Spherometer

screw, all the various points in connection with screws which were touched upon in Chapter II are to be remembered. It is generally found, however, that the chief uncertainty arises in the recognition of the event of the making of contact between the surface and the central leg. In using the three-legged instrument one of the outer legs is taken loosely between the finger and thumb, and the ease of rotation about the centre leg is estimated. It is found that there is a sudden change in the torque necessary to produce rotation when some of the weight of the spherometer begins to be supported by the central leg. By practice it is possible to acquire a fair consistency in the setting, so that the pressure on the central foot is the same when on the flat as when on the test curve. When this is the case the difference of the readings of the micrometer head should indicate accurately the displacement.

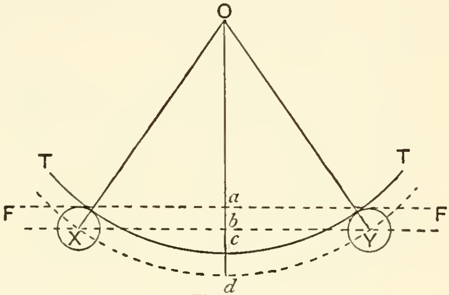

Fig. 107

Aldis Spherometer.—An improved spherometer employing a three-point support is constructed by Messrs. Pye & Sons, and is shown as modified by Guild in fig. 106. The test surfaces rest on three small spheres (steel ball bearings) mounted on a ring. Several sets of supports, giving circles of differing diameters, may be supplied with the instrument. The action of the ball supports will be understood by reference to fig. 107, in which X and Y are the centres of such supporting spheres, supposed to be placed at the extremities of a diameter of the standard circle, and the traces of the flat and test surfaces resting on them are indicated by FF and TT respectively. Let O be the centre of curvature of the test surface; with O as centre and OX as radius draw an arc which cuts the central line of symmetry in d. The remaining constructional lines will be understood from the figure. Let the radius of the test surface be R.

The sagitta measured by the instrument will be the interval ac. Since ab and cd are both equal to the radius r of the small spheres, $ac = bd$. This interval bd is, however, the sagitta which would be found if a surface of radius $R + r$ had been supported by true point supports at X and Y. The method of using the instrument is therefore to calculate the value of the radius (as found from the sagitta, and the mean radius of the standard circle through the centres of the supporting spheres), and then to deduct * the radius of these smaller spheres as found by measurement with micrometer calipers or by other means.

The distance between the auxiliary spheres is most readily found by a straightforward measurement with calipers, or a measuring microscope if greater accuracy is required.

In the instrument shown in fig. 106 the micrometer " head ", of diameter 2 in., is of the sleeve pattern and carries a vernier reading to 0·00001 in.† The test surface rests by gravity on the ball supports, and in the original instrument of Messrs. Pye is further kept in position by a small weight and plunger. In the original instrument the point of contact has to be estimated as before by the comparative ease of rotation.

Several objections to this method of setting can be raised. There is in reality a considerable deformation, comparable to the error of setting, of the parts in contact, before any considerable proportion of the weight is borne by the central screw. It is thus very desirable to provide some means by which such deformation can be avoided. Guild‡ has described a modification of the Aldis instrument in which a small ball of glass is mounted on the top of the central screw. When this is almost in contact with the flat or test surface, an interference system of Newton's rings is formed in the thin film of air between the surfaces, and this can be viewed, if a transparent piece of glass is being tested, with the aid of a suitably mounted microscope furnished with a vertical illuminator. In this manner, by watching the colour and relative size of the central spot of the ring system, it is possible to make settings of the greatest consistency, and the accuracy of the sagittal measurement is much improved. This enables the instrument to be used successfully for the testing of comparatively shallow surfaces. The method is not directly applicable to the testing of metallic surfaces. From the spreading of the centre spot when the weight of the lens is lifted by the glass surface the distortion of the contact surfaces may be estimated. Guild states that with the ordinary type of spherometer the magnitude of the error thus caused is of the order of 0·0004 mm., with the centre screw supporting a flat glass disc $\frac{1}{4}$ in. thick and weighing 35 gm. This error is overcome with the interference setting, made always so that the centre

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* Or *add*, if the test surface is concave.

† It is somewhat questionable whether the errors of the screw will not reach a considerably greater amount.

‡ *Trans. Opt. Soc.*, Jan., 1916.

black spot trisects the diameter of the first dark ring of the " Newton's rings " system. If the ball supports were made of the same size and material as that on the central plunger, the distortion arising at these points could also be determined.

Steinheil Spherometer.—In this instrument the principal feature is the mechanical device employed (as in the measuring machine described in Chapter III) to indicate the pressure between the central plunger and the test surface. Such an arrangement is shown diagrammatically in fig. 108. The central micrometer screw lifts a carriage which moves in slides. A lever *l* supports the central plunger *p*. The lever is hinged at *h* and the weight of the plunger is counterbalanced by a spring *s*. When the plunger is lifted into contact with the surface of the lens under test, the extra pressure causes a displacement of the end of the lever which moves over a scale. The reading of the micrometer screw has always to be taken for a definite scale reading of the lever. The pressure of the contact is thus standardized, and the arrangement permits of the use of a micrometer screw, which is, of course, very easy to use as far as the taking of readings is concerned. This latter point, viz. ease of reading, does not need consideration above other things where nothing but the highest accuracy is concerned, but in other cases the speed of obtaining results may be of considerable importance. The actual arrangement in the Steinheil instrument is much more refined than the simple arrangement shown in the diagram.

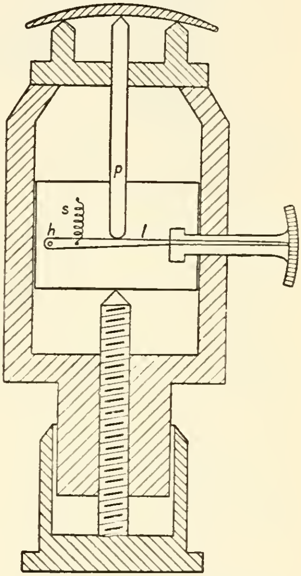

Fig. 108.—Spherometer with Mechanical Device for indicating Contact.

An alternative means of determining the sagittal displacement permits of the securing of a standardized pressure in the contact with much less difficulty, and this device will now be described.

The Abbe Spherometer.—In this instrument (fig. 109) the displacement of the central plunger is measured with the aid of a scale and reading microscope. A gentle upward tendency is given to the plunger by a counterbalancing weight with cords and pulleys, and thus the contact pressure is kept constant. The scale is divided to $\frac{1}{10}$ mm. and the reading microscope is held in a horizontal direction. The divided head of the micrometer eyepiece reads to 0·001 mm., and by estimation to 0·0001 mm. A careful use of the apparatus should make the error of the sagittal measurement less than 0·001 mm. *if the scale and micrometer can be trusted.*

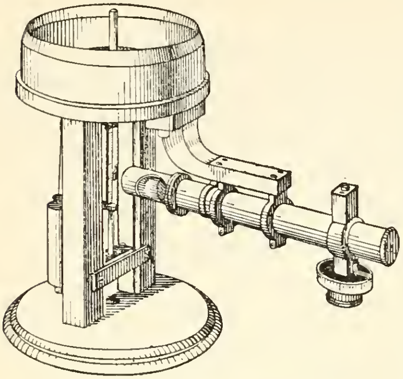

Fig. 109.—Abbe Spherometer

The lens rests on an accurately turned ring of hard gun-metal; several of such rings are supplied with each instrument for use with surfaces of varying radii and diameter. They are placed loosely on the face plate supported by the standards of the instrument. The section of each ring should be as shown (exaggerated) in fig. 110 (*a*) with a flat on the top, and the radius is thus different for concave and convex surfaces. In practice there is always a certain amount of rounding of the edge, and this, increasing with wear, will produce an effect resembling, on a very small scale, the result of the use of the ball supports (fig. 110 (*b*)). It is difficult to assess the amount of rounding or the radius of the edge, and thus the only thing which can be done is to keep the edges as perfect as possible.

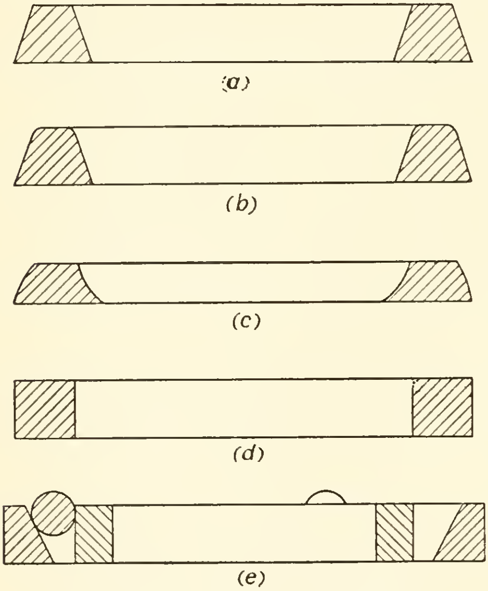

Fig. 110.—Forms of Support for Spherometers

The fitting of the plunger in the slides calls for great accuracy in construction, as, unless the distance between scale and microscope objective remains very accurately constant, there may arise " errors of runs ". All shake and wobble has therefore to be eliminated, and yet the motion has to be perfectly free in order that the pressure exerted by the counterbalancing weight may be fully transmitted. The difficulties of the mechanical construction have, however, been satisfactorily overcome by the makers.

Possible Improvements in Spherometers.—The main features of the problem will have been illustrated by means of the foregoing discussion. We may now notice certain suggestions which will still further improve the accuracy of the measurement. The improvement in the settings for the contact introduced by Guild's method is considerable. No elaborate apparatus is required and the micrometer screw can still be employed. The latter can probably be made to give readings as accurately as a scale with an eyepiece micrometer microscope, in so far as the absolute value of the results is concerned. It would not be hard to calibrate the screw both for absolute and periodic error if it were required to do so. On the other hand, Guild's arrangement is not suitable for use with an opaque curved object such as a grinding tool. Professor Conrady suggests that a plunger similar to that in the Abbe instrument could be given a gentle upward tendency by weights as before. This should carry a small glass plate at its lower end, and the remainder of the arrangements would be as in Guild's instrument; the " Newton's rings " system being formed between a glass ball (at the top of the screw) and the glass plate carried by the plunger. The arrangements are such that the ring system can be viewed with the aid of a lens or microscope. This seems to indicate a method of rendering the device more generally useful.

The remaining parts of the problem in relation to the sagittal measurement are all those involved in high-precision length measurements, by whatever means they are made.

As was previously indicated, however, the cause of greatest uncertainty when the shorter radii are to be measured is the radius of the standard circle. This is somewhat hard to measure very accurately in some of the instruments already described, and in some cases still harder to keep constant. Some suggestions due to Professor Conrady are of interest in this connection. The top edge of a ring could be ground on a flat tool. Fig. 110 (*c*) illustrates a ring in which the sharpness of the outer and inner edges can be brought up by grinding with suitable concave and convex spherical grinding tools respectively, the surfaces being portions of spheres. Fig. 110(*d*) shows a ring with truly cylindrical interior and exterior. The top surface can be ground down a little when necessary without changing the radii. The interior or exterior diameter would in this case be easy to measure very accurately if a measuring machine were available. Another suggestion of interest is due to Mr. H. W. Lee, of Messrs. Taylor, Taylor, & Hobson, Ltd., which is shown in fig. 110 (*e*). The balls are held in contact with the outer wall of a uniform cylinder. By determining, with measuring machine or calipers, the diameters of balls and cylinder, therefore, the radius of the standard circle can be determined with the utmost precision. No unsymmetrical microscope settings, such as may possibly be necessary when measuring up the mutual distances of three balls (fig. 105 (*b*)) are involved. The surface rests on the ideal form of support, which is better than having to rest on a ring.

In considering the relative advantages of these various arrangements, the possibility of absolute errors in scales or screws must be borne in mind. These often amount to values of the order of one or two thousandths of a millimetre in quite short lengths. It *may* be entirely in vain to try to push the accuracy of setting to great extremes if the calibration of scale or screw be neglected.

Dial Spherometer.—For quick measurements small instruments (widely used for testing the power of equiconvex spectacle lenses) can be obtained. They have a small ring, about 15 mm. in diameter, to be pressed into contact with the surface to be tested. A spring plunger actuates an index hand moving over a circular dial and registers curvature directly in terms of the reciprocal of the radius.

"Comparator" or "Substitution" Methods.—Messrs. W. Watson & Sons, Ltd., have made use of a noteworthy system for checking the curvatures of fairly deep concave surfaces. It is easy to grind a disc of glass so that the periphery possesses a definite radius which is measurable with accuracy. Moreover, by the aid of a suitable concave grinding tool the form of the edge may be made part of a sphere of practically the same radius which has its centre in the centre of the disc.

Such a disc when inserted like a template into a concave lens surface of the same radius ought to touch all along the edge. In practice, the area of contact and the width of small intervals between the surfaces can be estimated by watching the interferences formed in the air film with sodium light. With the aid of a fairly complete set of such discs and a measuring microscope for observing and measuring the interference bands, some very accurate curvature measurements can be made.

For convex surfaces of short radius it is possible to measure the curvature by comparison with a set of steel ball bearings in the manner which is illustrated in fig. 111. A microscope fitted with a vertical illuminator has, fixed on the stage, a diaphragm against which convex reflecting surfaces may be pressed. This serves as an automatic centring arrangement, but it is not essential for the smaller radii; it is enough to mount the reflecting surface co-axial with the microscope by any suitable means. A circular image of the illuminated aperture of the objective is formed by the reflecting surface, and a further image of this is then projected by the microscope, when suitably focused, into the focal plane of the eyepiece, where it may be measured by a scale or by a micrometer. By the use of a fairly complete set of the Hoffman steel balls (used for making up ball bearings) of which the diameter can be measured with great precision and which have surfaces of the greatest regularity, it is possible to calibrate the eyepiece scale in terms of the radius of curvature indicated by a given diameter of the disc image. For small radii the diameter is nearly proportional to the radius, so that interpolation is simple. A calibration curve is drawn. It is easy then to obtain the radius of any convex lens having radii between 3 in. and $\frac{1}{32}$ in. For fuller details the original paper * by Guild should be consulted. There are other devices † too numerous to mention for measurements of this kind.

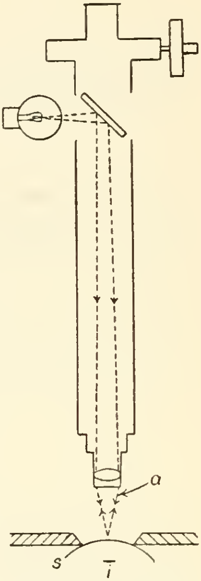

Fig. 111.—Curvature Measurement by Guild's Substitution Method

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* Guild, *Trans. Opt. Soc.*, Vol. XXII, No. 3.

† *Practical Optics for the Laboratory and Workshop*, B. K. Johnson, p. 85.

CHAPTER XI

Instruments for the Measurement of Focal Length

Theoretical Summary.—Before proceeding directly to the description of the several instruments it may be useful to give a short summary of the theory of the subject treated from the practical standpoint. The theory of most focometric measurements can be easily rederived, if forgotten, from the usual geometrical constructions giving the courses of rays in centred optical systems. These are concerned, of course, with paraxial rays only for uncorrected systems, but they also hold very nearly for other rays in corrected lenses with which practical measurements are usually concerned.

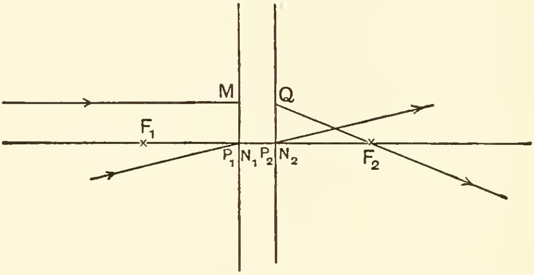

Fig. 112

The chief features of the Gaussian first-order theory may be summarized as follows:

1. Bundles of rays parallel to the axis, after passing through the lens system, will converge to or diverge from a point; this is true for rays in the opposite direction, the point being of course different. F_1 and F_2 (fig. 112) represent these two points for the positive system depicted. Thus all rays parallel to the axis, and passing from left to right, will intersect in the point F_2.

2. There are two principal points (P_1, P_2) for the system. An object at one point has an *erect* image of the same size at the other.

Planes perpendicular to the axis passing through these points are the principal planes. A ray whose path before refraction (produced if necessary) would cut one principal plane in the point M has such a path after refraction that its path (produced if necessary) cuts the other principal plane in the point Q at the same distance from the axis.

3. There are two nodal points, N_1 and N_2; a ray directed to one of these leaves the system as though it came from the other in a parallel direction: they are thus conjugate foci, just as are the principal points. If the initial and final media are the same the nodal and principal points coincide.

4. If the initial and final media are the same, the focal lengths of the system, i.e. the distances F_1P_1 and P_2F_2 are equal.

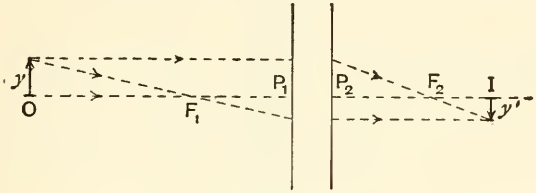

Fig. 113

Inset is a typical example (fig. 113) for the construction to find the image of an object at O. The "height" of the object perpendicular to the axis is y, the corresponding size of the image is y'.

The following relations are useful for a system in which the initial and final media are the same:

Let the distance from the first principal focus F_1 to the object be x; i.e. $F_1O = x$.

Let the distance from the second principal focus F_2 to the image be x'; i.e. $F_2I = x'$. Also let $F_1P_1 = P_2F_2 = f$.

Then $$xx' = -f^2, \qquad \text{(1)}$$

and $$\frac{y'}{y} = \frac{f}{x}, \qquad \text{(2)}$$

where distances are positive from left to right* (and *above* the axis) when the light passes from left to right.

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* E. T. Whittaker, *Theory of Optical Instruments*, p. 11.

The formula (2) is at once reobtained from fig. 113 if the construction be remembered, and also the corresponding relation $\frac{y}{y'} = -\frac{f}{x'}$. Equation (1) follows from the product of these two.

In the case above x and x' have opposite signs. f^2 is always positive, of course, but in the above case f itself is positive. x is negative, therefore y' and y have opposite signs (the image is inverted).

A typical construction for a divergent system is also given (fig. 114). Then as before

$$F_1O.F_2I = -f^2,$$

and

$$\frac{y'}{y} = \frac{F_1P_1}{F_1O} = \frac{f}{x}.$$

In this case f is negative, x is negative, and both F_1P_1 and F_1O are negative.

Before going on to apply these formulæ to the measurement of focal length it will be well to examine the properties of a few representative lenses and systems to see how far they comply with the properties which we should assume in using any particular method, for it must be realized that very different methods have to be adopted when working with, say, a telescope objective and a microscope objective; in fact, when discussing the "focal length" of a microscope objective the use of the term tends to be somewhat misleading, because in this case a perfect image could not be obtained at the principal focus on account of spherical aberration, and an effort to measure the focal length by finding the focus for parallel light would be unwise on this account if for no other reasons. The quantity required has the dimensions of a length, as will be seen from equation (2) above. A concrete conception is that of the focal length of an indefinitely thin lens, being the actual distance of the principal focus from the vortex of the lens surface. The focal length of a complex system may then be considered as that of the equivalent thin lens which will produce an image of the same size as is produced by the

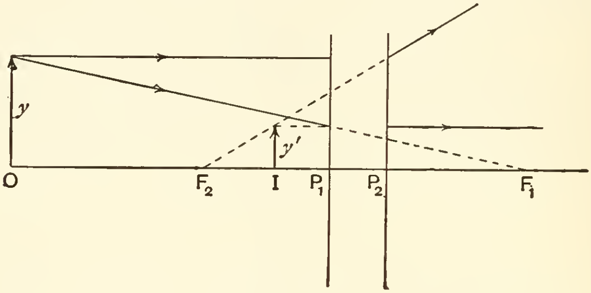

Fig. 114

complex system, the object being considered to be at an indefinitely great distance. In so far as the system fails to yield images without aberration at the principal focus or at other image points, its " focal length " becomes less definite and only possesses a precise meaning when measurements are made with reference to particularly specified ray paths. Its definition must then rest (for our purposes at least) on the equations given above. It will be seen that the whole conception is intimately bound up with ideas of magnification; the very definition of the principal points is dependent on the unit magnification corresponding to these conjugate foci. This being the case, it is natural that most methods of determining focal length should be based on measurements of magnification,* except in the case of thin lenses, where an approximate measurement of back focusing distance is all that is required.

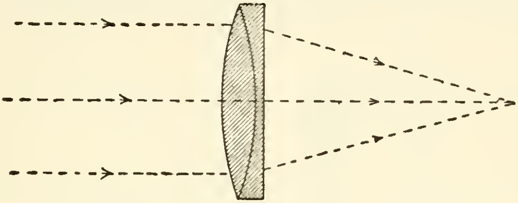

Fig. 115.—The Telescope Object Glass (diagrammatic)

1. *The Telescope Object Glass.*—Such a lens often consists of a doublet, crown and flint. Its chromatic aberration is corrected for the axis sufficiently nearly, and the spherical aberration is so corrected by choice of curves, or figuring, that for one wave-length at least we may say with sufficient nearness, when dealing with the topic from the point of view of focometry, that all rays parallel to the axis are brought to a point (fig. 115).

With regard to oblique pencils of parallel rays, the object glass is generally made to fulfil the sine condition (p. 192) which, as Professor Cheshire points out, necessitates that, instead of having a principal *plane* where the paths of incident and refracted rays meet, we have to deal with a spherical principal surface.† The fulfilment of the sine condition secures the good definition of the image at points just off the axis. In any simple telescope object glass, however, astigmatism and curvature of field become increasingly prominent at a few degrees from the axis. Hence it will be seen that no method must be used which would tend to make the lens act artificially; if the sizes of images are measured, they must be taken when produced by the objective working at its proper focal distance, and measurements of the image should be made near the axis only. We can, however, sometimes make measurements on particular "pencils" which follow paths only approximately similar to those which would be followed in the normal action of the system (Abbe focometer, p. 189).

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* Almost the only alternative is the method given by Gauss (*Gottingen Abhandl.*, Vol. I, 1838–41, pp. 1–20), where Gauss shows how to calculate *f* from two measured displacements of the object point, and the two resulting displacements of the image point. The necessary equations are easily derived from equation (1) above.

† See Cheshire, *Jour. Queckett Club*, 1904; *Trans. Optical Society*, May, 1917.

2. *The Microscope Objective.*—Microscope objectives act under very different conditions from telescope object glasses, as they receive strongly divergent instead of nearly parallel light (see fig. 116).

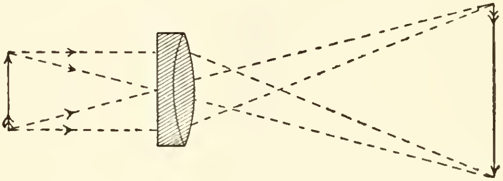


Fig. 116.—The Microscope Objective (diagrammatic)

Generally speaking, they are also designed for use at a particular working distance, for spherical aberration rapidly becomes prominent otherwise. The sine condition is fulfilled, but some little distortion may be found in the image formed by a complex lens. This means that if we base one determination of focal length on the magnification of the image of a very small object and another on the magnification of a larger object we may expect to find a slight discrepancy between the two results. It is seldom, however, that focal length need be measured to a greater accuracy than, say, 1 part in 100, and it is not likely that the presence of such slight distortion as is usually found in microscope objectives will materially affect the results.

3. *The Photographic Objective.*—A good photographic objective is the lens which most nearly fulfils the conditions of an ideal system possessing flat principal planes and flat focal planes. The axial definition is not so good as in the case of the telescope or microscope objective, as a considerable amount of spherical aberration has usually to be tolerated in order to correct the outer parts of the field; but for most practical purposes it is quite sufficient to give an exact meaning to the term "focal length", and we have a great deal

more liberty in using large objects and rays at larger angles with the axis. Moreover, a good lens will give sufficiently good definition for almost any positions of object and image. We may therefore even use virtual image methods if it is so desired. These remarks apply with reservation only to the simple type of landscape lenses where distortion is fairly prominent.

4. *Eyepieces.*—With regard to these it may be noted that, unlike microscope objectives, which produce a real image at a finite distance, the eyepiece produces a virtual image at an infinite distance; in other words, the rays from a particular point of the image which the eyepiece is used to examine should be parallel in emergence from the eyelens. The separation of the lenses is so chosen that achromatism in this sense is obtained, rays of all colours being rendered nearly parallel (fig. 117). Hence if we wish to make measurements on an eyepiece we should, as far as possible, try to imitate the condition of bringing parallel light *in at the eyelens* and examining the final image. There are, of course, other aberrations, noticeably distortion, which are often very prominent in eyepieces.

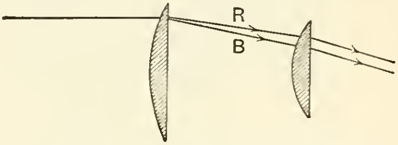


Fig. 117.—A Simple Huygenian Eyepiece (diagrammatic)

The Optical Bench.—So many patterns of optical bench are extant that the present treatment will be confined to a brief description of one type, and its accessories, which has been developed in the optical engineering department at the Imperial College of Science and Technology, and which has been found convenient for the various purposes of an optical laboratory.

The base consists of a 2-m. length of steel shafting held parallel to a planed bar at 3 in. distance. The form and position of the supports, furnished with levelling screws, will be seen in fig. 118. A number of stands, with feet suitably shaped to slide on the bench, are provided. They can be clamped under the slides if desired. It would be a simple matter to adopt the geometric mounting described in Chapter I to the present purpose. The stands are of uniform shape and are provided with a fiducial mark or a vernier to slide over the scale engraved on the cylinder. It suffices to read to 0·1 mm. The vertical pillars of the stands are drilled with a hole which is of a standard size for the laboratory, so that the holders which they receive can be interchanged, and any apparatus from this bench can be used on projection benches, &c. The holders are arranged for: ground-glass screens; correction rod (20 cm.); grainless photographic scales of 5 cm. (divided into millimetres) on glass by *Rheinberg*; eyepiece (monocentric or Ramsden) containing in its focal plane a scale or graticule also by Rheinberg (1 cm. divided into 100 parts); a screw micrometer eyepiece; sundry diaphragms, including one which is a vertical slit 1 mm. in width; a lamp with frosted or opal bulb; sundry clamps; sundry lens holders and holders for mirrors and other optical parts (some of these will be described in the next paragraph); a microscope with axis parallel to the slides and otherwise adjustable, and having a separate focusing motion, also with a Rheinberg graticule in its focal plane. With regard to the lens-holders, the simplest form, which is very useful, is a plain wooden " V "; this will hold " edged " lenses well with the aid of a little wax. Much more ambitious is the ring, with three radial plungers which can be clamped in any position, shown in fig. 119 (*a*).

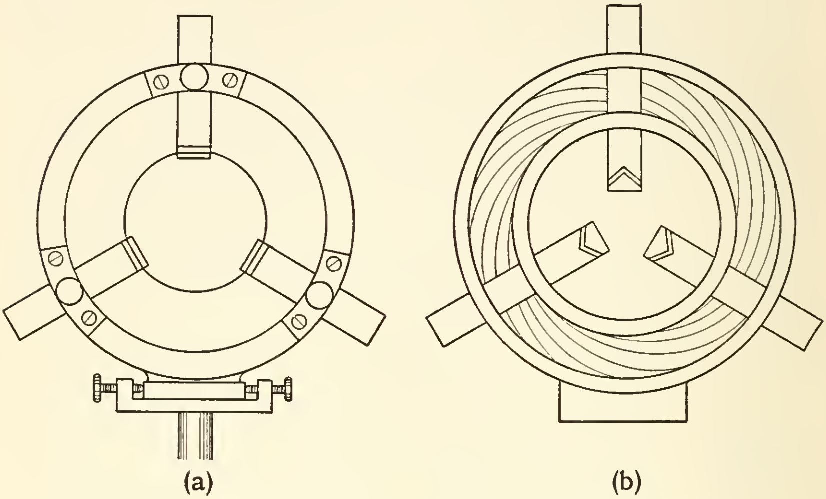
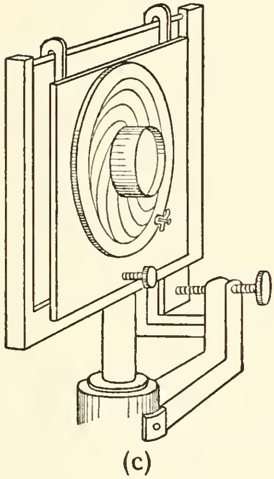


Fig. 119.—Types of Mount for Lenses useful on Optical Bench



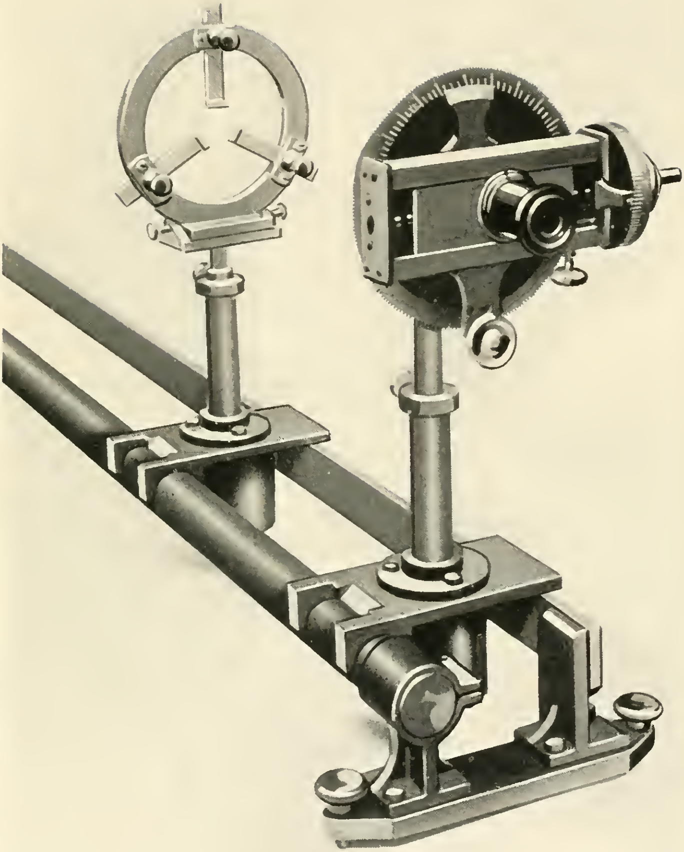


Fig. 118.—Optical Bench

The whole ring can be given a lateral movement for centring purposes, and the plungers are provided with a flange at the end to hold the lens or other object more securely. A device employed at the Physical Laboratory in Rome is also shown diagrammatically in fig. 119 (*b*). Teeth are cut on the plungers so that they can be given a radial motion by the rotation of a ring bearing spirally-cut teeth which engage those of the plungers in a manner similar to the self-centring chuck. Another device is also shown in fig. 119 (*c*). This consists of a large iris diaphragm of massive construction in which the leaves are sufficiently strong to grip and hold lenses up to a diameter of 3½ in. The leaves can be clamped in any desired position. For convenience in adjustment the whole diaphragm is hooked to the top bar of a " gallows " mount, the design of which is due to Professor Cheshire. The gallows consists of a square frame supported by a rod fitting the optical bench holders. The top of the frame is a circular rod over which the hooks are made to fit. Small angular adjustments in altitude can be given to the diaphragm as its lower part bears, by the force of gravity, against the nose of a screw passing through the lower bar of the frame. Alternatively, the screw may pass through the swinging member. A screw adjustment for azimuth can also be provided if necessary.

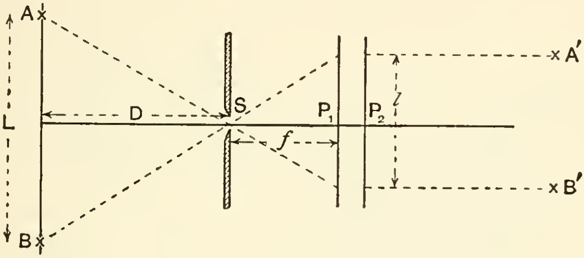


Fig. 120

The bench and fittings described above will fulfil most of the needs for optical testing work. Nodal slide measurements (see a later paragraph) are often made on a separate bench.

Determination of Focal Length by an Optical Bench Method.—A number of " bench " methods for the determination of the focal length of " thick " lenses are described in text-books of practical physics, but one quick method due to Professor Cheshire may be described here. The plan diagram (fig. 120) illustrates the principle, which is as follows. A slit S is first placed at the principal focus of the test lens, the principal planes P_1, P_2 of which are shown. This is done by placing a lamp to the left of S and a plane silvered mirror to the right of the lens, when an "auto-collimation" image of the slit may be focused beside it. In practice the slit may be about 3 to 5 mm. wide. The lamp and mirror are then removed, when the lens forms the images A′ and B′ of two divisions A and B on a scale to the left of S. The lengths A′B′ and D as shown in the figure can then be ascertained; A′B′ by means of the micrometer eyepiece, and D with the aid of the correction rod. As L is known, the focal length of the lens then follows from the relation below (the derivation of which is obvious from the Gaussian diagram),

$$\frac{f}{l} = \frac{D}{L},$$

This is a typical example of the use of the telecentric stop* in focometry. In order to make the setting more accurate, cross hairs might be stretched across the slit and viewed, together with the auto-collimated images, by a microscope which carries a plane glass reflector to illuminate the cross wires. There is no occasion to make the slit *very* narrow, as the telecentric action will not then give an advantage sufficiently great to compensate for the loss of definition owing to the restriction of the effective aperture of the lens. The scale may be represented simply by a wooden bar carrying two marks A and B, but in any case the scale or bar should carry a bold horizontal line which is thus perpendicular to the length of the slit. The image of this line is viewed by the micrometer eyepiece and serves for focusing the image. This focusing is easy and exact, as the whole aperture of the lens is available for giving clear definition to the image of a horizontal line. The important points are the accurate location of the principal focus and the accurate measurement of the magnification.

Rotating Table Methods.—An extremely useful instrument for the focometry of telescope object glasses is illustrated in fig. 121. In practice it is conveniently constructed from the "table" of a theodolite without the usual upper parts. The table is fitted with a pillar bored to receive the optical bench fittings previously described. A special device is provided by means of which a straight Chesterman steel scale can be supported horizontally, and this forms a miniature optical bench carrying simple slotted fittings to support any necessary optical parts. For the present purpose a small "gallows type" lens-holder and an eyepiece with a scale in its focal plane are required. The test lens, of which the principal planes P_1, P_2 are indicated in fig. 121 (*b*), is mounted on the bench and is made to form, near one end of the eyepiece scale S, an image of a *very distant* object, a flagstaff, for example. The focus is made as exact as possible. On rotating the table the image moves a distance d across the scale, and by means of the verniers the angular interval α, corresponding to its passage over a definite number of scale divisions, may be found. Then from the geometry of the Gaussian diagram:

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* P. 23.

$$f = \frac{d}{2} \cdot \frac{1}{\tan\frac{\alpha}{2}}.$$

Since, however, the angular rotation will not exceed more than 2° or 3° of arc, it will be sufficiently accurate to calculate by the formula

$$f = \frac{d}{\tan\alpha}.$$

The verniers of the circle should read to 5″ or 10″ of arc. It may be of interest to note that if an object, so close as to displace the image plane perceptibly from the true focal plane, had to be employed, some diminution of the inaccuracy thus caused might be obtained by the use of a suitable telecentric stop at the front principal focus of the test lens, and having this focus at the centre of rotation.

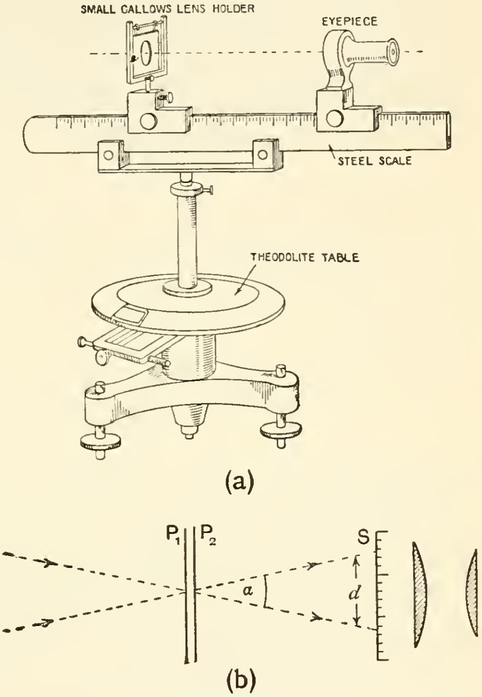

Fig. 121.—Apparatus for the Rotating Table Method

Another alternative is to mount a collimator (an arrangement with a cross wire supported accurately in the principal focus of a telescope object glass) on the theodolite table, and to measure the displacement of the image of the collimator cross wires formed by the test lens for a measured rotation of the collimator. It will be seen that if two beams of "parallel" light making a definite angle with each other (such, for example, as are derived from two stars) are available, the measurement of focal length of a suitable lens is a comparatively easy one, as the displacement of the image has only to be multiplied by a constant in order to evaluate the focal length. The two beams are conveniently derived in the laboratory by a "focal collimator" in which two cross lines are placed in the focal plane of a good telescope object glass. The focal length of a test lens is found by using it together with the focal collimator on the optical bench. The corresponding Gaussian diagram is shown in fig. 122. If then the interval between the images a' and b' of the cross wires a and b is measured, the focal length of the test lens is found at once by multiplying this interval by a constant. This will be more readily understood by reference to the geometry of the diagram. The collimator may be considered to yield two parallel beams of light at a definite angle. When these meet the test lens symmetrically the size of the image is independent of the distance of the test lens from the collimator. We may therefore, for convenience, take the case in which the posterior principal focus of the collimator lens coincides with the anterior principal focus of the test lens. Let the focal lengths of collimator and test lens be f_c and f_t respectively, and let their common focus be at O in fig. 122. Then from the Gaussian theory

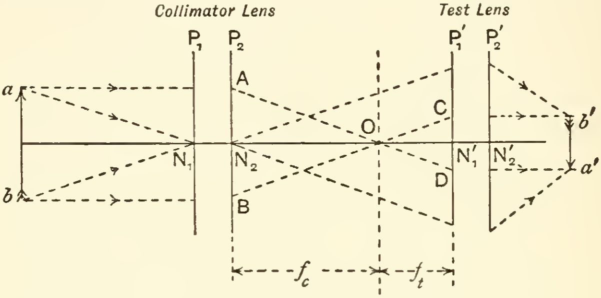

Fig. 122.—Use of Focal Collimator

$$f_t = \frac{f_c}{ab} \,.\, a'b'.$$

The quantity $\frac{f_c}{ab}$ is the constant to which reference was made. A convenient focal length for the objective in such an instrument intended for optical bench work is about 20 cm. If then the cross wires are 4 mm. apart the so-called constant of the instrument is 50.

We may quote Professor Cheshire's published description* of the method of adjusting and determining the constant of such a foco-collimator.

" (1) A graticule is taken with a pair of scale lines of an unknown but convenient distance apart. This graticule is carefully adjusted in the focal plane of the collimator lens by standing the collimator vertically upon a plane reflector and focusing the graticule with a coaxial reading microscope. A sheet of thin mica is then interposed and tilted between the object of the microscope and the graticule to throw light down upon the graticule and thence to the mirror. The image of the graticule scale then appears in or near to the plane of the scale itself. The latter is adjusted up or down until the object scale and its image are seen simultaneously in focus.

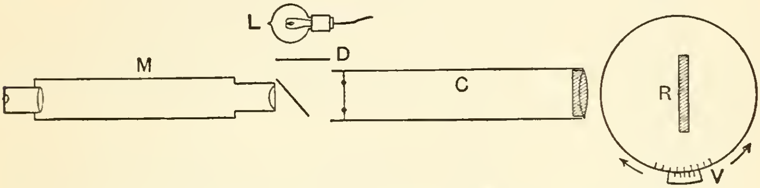

Fig. 123.—Adjustment and Determination of Constant of Focal Collimator

C, Collimator. D, Diffusing screen. L, Lamp. M, Microscope. R, Plane mirror. V, Circle and vernier.

" (2) The angle subtended by the scale division is determined practically with the same auto-collimating apparatus (fig. 123). The mirror, however, is mounted vertically on an accurately divided horizontal table, and is so adjusted initially that the left of the two line images is superposed on the right line of the graticule. The graduated table is then rotated until the right line of the image is superposed upon the left of the graticule. The angle through which the table is rotated to produce this displacement is the angle subtended by the two scale lines at the principal point of the collimator lens, and this angle being small its reciprocal is the number by which the length of the image of the two scale lines projected by any lens must be multiplied to obtain the focal length of that lens. This constant multiplier should be engraved upon the foco-collimator."

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* *Trans. Opt. Soc.*, Vol. XXII, No. 1, p. 29.

It has been thought worth while to quote this clear description *in extenso* because the more extensive adoption of instruments of this kind for practical focometry is most desirable. Furthermore, even in teaching laboratories, the construction and adjustment of such apparatus would afford a far better exercise in optical practice than the usual methods of determining the focal lengths of " thick " lenses, an operation which seems to be regarded by many physics students as extremely obscure and difficult.

If a good circle with microscopes or verniers reading to 10″ is not available for mounting the mirror separately, a theodolite can be used to determine the angular subtense of the divisions through the collimator lens. Otherwise the distance between the marks may be measured with a measuring microscope, and the focal length of the lens can be separately found; the angle could then be calculated.

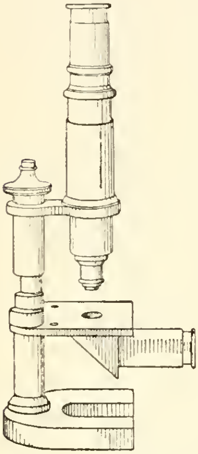

Fig. 124.—Cheshire Focometer

The Cheshire Focometer.— Smaller collimators may be employed for use with shorter focus lenses, such as eyepieces or microscope objectives, as has been suggested by Blakesley. Professor Cheshire has described* a convenient focometer constructed on this principle, the action of which is shown diagrammatically in fig. 122. The complete instrument is shown in fig. 124. Referring to the diagram (fig. 122), the collimator is seen on the left, and the two parallel beams of light which are derived from it are indicated. A telecentric stop is placed as nearly as possible in the second principal focus O of the collimator lens; the object of the stop is to minimize the effect of any positional error of the cross wires. The test lens (with its principal focus at the stop) forms an image which has to be measured as before. With short-focus systems, however, the image is very small, and has to be measured by means of a microscope with a scale in its focal plane. (Compare the magnification methods described in Chapter III.) Let y be the distance between the cross wires a and b in the collimator, let f_c and f_t be the focal lengths of the collimator lens and test lens respectively, and let y' be the interval $a'b'$ between the images of a and b. Then, as before,

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* *Jour. Roy. Microscopical Soc.*, 1914, pp. 513–9.

$$a'b' = \frac{f_t}{f_c} . ab.$$

Let M be the magnification of the microscope and let $a''b''$ be the size, as measured on the microscope scale, of the final image. Then

$$\frac{a''b''}{\mathrm{M}} = \frac{f_t}{f_c} . ab, \text{ and } f_t = a''b'' . \frac{f_c}{\mathrm{M}.ab}.$$

If the microscope scale be in millimetres it is not difficult to make the instrument direct reading by arranging that $\frac{f_c}{\mathrm{M}.ab}$ shall be equal to 10, by adjusting the dimensions of *ab* and the value of M. The stop, in this case a slit 3 or 4 mm. wide parallel to the cross lines in the collimator, causes a considerable depth of focus, as far as the significant cross lines are concerned, on account of the limited aperture of the lens at right angles to their direction. The whole diameter of the aperture, however, is employed to focus the image of a *horizontal* cross wire, always placed in the diaphragm at right angles to the pair of cross lines mentioned above, and it is on this that the microscope is focused before taking a reading. It is to be noticed that when the auxiliary microscope is employed to measure the image, the telecentric action of the stop is no longer effective in regulating the size of the final image.

In practice the collimator is made horizontal, and, by the aid of a right-angled prism, the light is then reflected vertically upwards along the axis of a microscope. The telecentric stop T is in the plane of the horizontal stage and the test lens is rested on it; the microscope is then racked up or down until the image is found, when the focal length of the test lens may be read off from the scale.

The Abbe Focometer (Zeiss).—This instrument aims at a somewhat greater precision in certain classes of focometric measurements than the majority of the methods previously indicated; the makers claim that an accuracy of 1 part in 1000 can be obtained. As usually constructed and supplied, its *ordinary* use is confined to positive systems having focal lengths greater than about 10 cm., or any negative system. The dimensions of the system must be small enough to enable it to be placed on the table of the instrument, and should not exceed 10 cm. in diameter or 5 cm. in thickness.

With the present instrument the determination of magnification is, by the application of a telecentric principle, made more independent of the location of images. Also the magnifications found are really those which apply for definite zones in the circular aperture of the test lens. In general the lens under test is made to form virtual images of two scales placed perpendicular to its axis at a distance apart determinable with accuracy. These images may be observed by a coaxial telescope with an objective of suitable power. The lens can then be given a measured displacement at right angles to the axis, and the deviation of the light passing through various zones of the lens can then be found by suitable observations on the scales. The focal length of the system can thus be calculated. Fig. 125 shows a view of the instrument, and fig. 126 will illustrate the theory of the measurement.

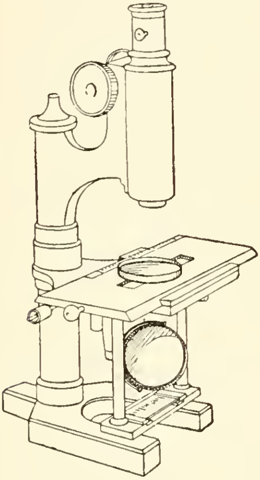

Fig. 125.—Abbe Focometer

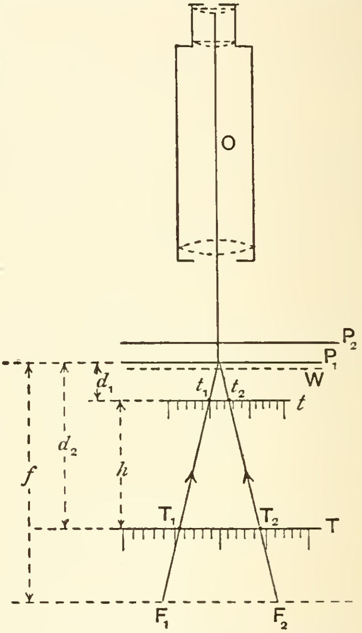

Fig. 126.—Optical Principles of Zeiss-Abbe Focometer

The optical pointer O* is supported at right angles to the direction of the slide W and the two scales t and T, which are separated by a distance of about 10 cm. The upper scale can be swung out of its central position when required. In one instrument the scale and vernier register the motion of the slide to 0·002 cm., and the scales t and T are divided to 0·01 cm. and 0·05 cm. respectively.

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* For the meaning of this expression, i.e. " optical pointer ", see p. 17.

The lens is supported on the slide W; it forms virtual images of the two scales at greater distances, and in order to focus these the optical pointer is provided with a series of objectives of varying focal lengths, which, with the aid of the rack motion of the tube, will do all that is required for any image position farther from the objective than the scale t. Assuming that the slide moves the test lens system at right angles to the axis of the optical pointer, and that the axis of the test lens is coincident with that of the microscope when the lens is in its central position, it will be seen that movement of the horizontal slide will always keep the principal planes of the lens in the same horizontal loci. Let F_1 and F_2 be the two positions of the principal focus of the test lens. Then since the axial ray through the microscope is always parallel to the axis of the test lens, it will pass through F_1 and F_2 respectively in the two positions of the lens; and, further, by the use of suitable objectives the divisions on the scales at the points intersected by the ray can be focused on the central cross wire of the microscope or pointer. Hence, for a given displacement F_1F_2 of the lens, the corresponding displacements t_1t_2 or T_1T_2 on the scales could be found by watching the passage of the images over the cross wire in the microscope. In this way we may find, in separate operations,

$$\frac{T_1T_2}{F_1F_2} = \frac{d_2}{f} = \alpha \text{ (say)},$$

and

$$\frac{t_1t_2}{F_1F_2} = \frac{d_1}{f} = \beta \text{ (say)},$$

for a particular zone (of diameter $= F_1F_2$) of the lens, where f is the focal length of the lens and d_1, d_2 are the distances of the two scales from the lower principal plane.

There is evidently no need to take exactly the same displacement F_1F_2 in each case; in practice the images of a convenient whole number of scale divisions is allowed to traverse the cross wire, and the corresponding displacement is found from the vernier on the horizontal slide.

Subtracting the above equations we obtain

$$\frac{d_2 - d_1}{f} = \alpha - \beta,$$

but $d_2 - d_1$ is measurable on the apparatus, a convenient depth gauge being supplied for the purpose. Therefore f can be determined, as α and β have been measured. In a lens which fulfils the sine condition* for an object at an infinite distance, the second principal "plane" is really a sphere with f as radius, where f is the paraxial focal length. This is shown in section in fig. 127. If the displacement on each side of the centre of the lens on the Abbe focometer is h, say, the focal length as found by the instrument will be equal to $f \cos u'$ where $\sin u' = \frac{h}{f}$. It should be possible, therefore, to check the correction of the lens by this means. Measurements can be made for a number of zones, starting from one near the centre. Divergences of focal length from those given by $f \cos u'$ can be taken, if systematic, to indicate the presence of aberration. It is advisable to work with monochromatic light.

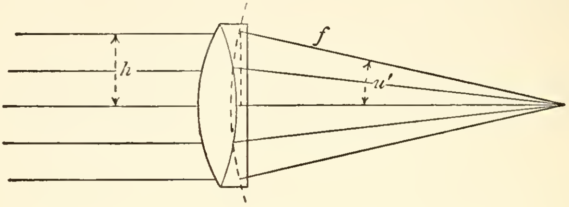

Fig. 127

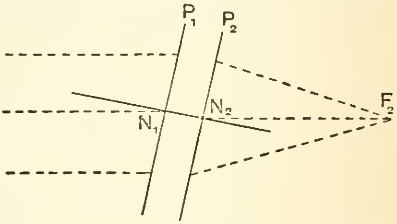

Fig. 128

The Nodal Slide.—Referring to fig. 128, let P_1 and P_2 be the principal planes of a convergent optical system, and N_1, N_2 the cor-

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[*] The "sine condition", which has to be fulfilled by lenses when coma is eliminated, relates to the angles u and u', with the axis made by rays diverging from an axial object and converging to the corresponding image respectively. It is expressed by the equation $\frac{\sin u}{\sin u'} = M =$ constant, where M is the ratio of the linear dimensions of a small axial image to those of the small axial object. The value of M, and consequently the ratio of $\sin u$ to $\sin u'$, must be constant, no matter at what height the ray intersects the lens. When the incident light is parallel the condition becomes $\frac{h}{\sin u'} =$ constant, where h is the distance from the axis at which the ray cuts the lens.

responding nodal points. It is assumed that the initial and final media have the same respective index, so that principal and nodal points coincide. Let parallel light be incident on the system, which is rotated about the point N_2 so that the optical axis of the lens swings through a small angular range above and below the line N_2F_2. The ray from N_2 remains always parallel to its original direction, and therefore it may be concluded that the position of the image, always found somewhere on the path of this ray, *will not suffer lateral displacement with such an angular movement of the lens about this one definite point.*

This property is employed in the focometric apparatus known as the nodal slide (fig. 129). A substantial fitting F is mounted on the optical bench and carries an upper part M rotating on a vertical axis. A special pointer P indicates on the bench scale the relative position of this axis. The part M carries in its turn a member N which is moved relatively to M by means of a horizontal rack and pinion. N is provided with a lens-holder H so that the lens L may be held with its axis horizontal.

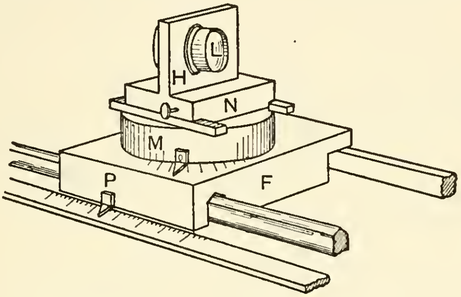

Fig. 129.—Simple Form of Nodal Slide

The optical bench also carries a collimator, with a fine cross wire or pinhole at its principal focus to provide the necessary parallel light, and also, on the further side of the lens, a microscope to observe the image produced by the lens system under test.

Means have to be provided for the location of the anterior focus of the microscope relatively to the bench scale. The focus is usually located by a " stage " carrying cross wires with a corresponding pointer. The experimental procedure is simply to vary the position of the lens relatively to the axis of rotation until the image is stationary on rotation of the lens; the focal length then follows from the bench readings of the above axis and the anterior focus of the microscope.

A convenient form of nodal slide is embodied by Messrs. R. and J. Beck, Ltd., in their testing bench for photographic lenses.

Difficulty arises in the case of telephoto systems where the nodal points may be very unsymmetrically placed with regard to the system. Smith and Anderson* discuss the question and point out that a small angular rotation may in some cases displace the aperture of the lens so far from its central position that it begins to be filled only partially by light from the collimator. Some possible arrangements for giving suitable parallel displacements to the collimator are discussed. A possible method for securing two accurately parallel beams is to employ a single collimator, and a pentagonal prism (or a pair of mirrors at 45°) mounted on a suitable parallel sliding arrangement. Reference may be made to the description of range-finder adjusters, Chapter X, p. 122.

The nodal slide is a "null" method. It is a convenient mechanical means of determining that point on the axis of the lens at which the angular subtense of the image displacement in the focal plane is equal to the angular displacement of the incident light. In discussing the rotating table method we derived an equation for the focal length. We now have a convenient mechanical means of finding the solution without observing any numerical results. It is therefore essentially a "magnification" method.

The chief difficulty in the method is the mechanical one that it is difficult to secure that the mechanical axis of rotation and the optical axis of the lens shall always intersect. Experienced workers prefer other slower, but more sure, magnification methods.

Zeiss Focometer.†—Unlike the instruments previously described, this focometer is only suitable for determining the distance of the principal focus from the surface of a lens. Fig. 130 gives a diagram of its construction. C is a vertical collimator with perpendicular cross wires at the focus. L is the lens under test, which rests on an adjustable collar A so that its lower surface is just in contact with a small fixed stop K. A mirror or right-angled prism reflects the light into a horizontal direction and it is brought finally to a focus by the lens S, which is placed so that its anterior principal focus coincides with the fixed stop K, and therefore with the surface of the test lens. The final image is viewed by means of a microscope T which moves over a scale; by this means any displacement of the image in a direction parallel to the axis may be measured. In fig. 131 the optical action of the instrument is shown.

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* *Trans. Opt. Soc.*, Vol. XXIII, No. 3, p. 188.

† Patent specification, 12100/1914.

The image is first found by the microscope before the test lens is placed in position, and is located at P, the second or posterior principal focus of S. When the lens L is introduced the image is shifted to O′.

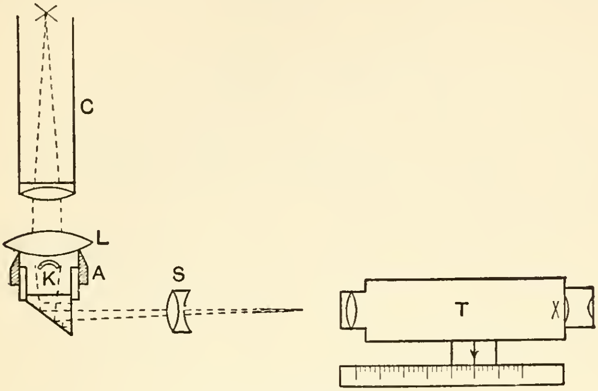

Fig. 130.—Zeiss Focometer

The formula given on p. 177, $xx' = -f^2$, may now be applied to the lens S. x is the distance from the first principal focus to the object, i.e. to the image at O which would be formed by L in the absence of S; x' is the distance from the second principal focus to the image. The relation becomes $KO.PO' = -f^2$, or $KO = \frac{\text{constant}}{PO'}$. The length PO′ is found from the displacement of the microscope; it is inversely proportional to the distance of the second principal focus of the lens L from its back surface. For a thin spectacle lens it is sufficient therefore to take the displacement as proportional to the power of the lens. Since the collimator C carries cross wires mutually at right angles, it is possible to measure the power of cylindrical or sphero-cylindrical lenses by turning the cross-wire frame until the directions of the maximum and minimum

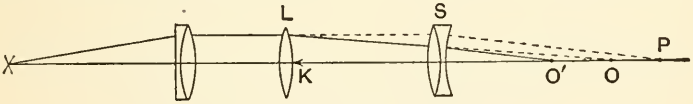

Fig. 131.—Optical Theory of Zeiss Focometer

powers have been found. A circular scale is provided on the mount of the cross wires for this purpose.

General Conclusions.—It is not often that focal lengths need to be measured with an accuracy better than a 1 per cent error allows. The present account has given one good optical bench method for medium focal lengths. Focal collimators for medium and short focal lengths should be used in every optical laboratory, but an elaborate instrument like the Abbe focometer will find comparatively little use, as the determination even of a single focal length is a laborious process. For teaching, of course, one good method without very special apparatus ought to be adopted for each class of optical system.

CHAPTER XII

Photometric Instruments

General Principles of Instruments depending on Photometric Estimations.—A great number of instruments, besides photometers as such, depend on photometric estimations for their operation. Spectrophotometers, colorimeters, optical pyrometers, and polarimeters of many types are similar in so far as they depend on the power of the eye to recognize, under suitable conditions, the equality of brightness of uniformly illuminated areas or objects in the visual field of view, provided that when a brightness equality has been obtained the colour of the areas is identical.

The conditions for maximum sensitiveness require, firstly, that the areas shall be contiguous so that (at any rate in the simpler types of instruments) the line of separation tends to disappear when the match is perfect; secondly, that the matching field shall subtend an area of at least 2° in the visual field and shall be larger if possible; thirdly, that work shall be done between certain values of absolute brightness. Departures from some of these conditions are sometimes inevitable, but the first and second points should be provided for in the design of an instrument, and the second should be considered far more often during actual work. These points are of such importance that they may with advantage be discussed more fully.

Contiguity of Areas in the Matching Field.—There is a general consensus of opinion that the sensitiveness to contrast is greatly diminished when the contrasting patches are separated. There is some evidence, however, to show that with sufficient practice a very considerable accuracy may be acquired even when working with patches separated by three or four times the width of each patch. Persons accustomed to work with the *Tintometer*, a colour-matching instrument in which the patches are so separated, acquire the power of detecting contrasts almost as small as those which can be recognized with contiguous areas. It is sometimes objected that the tendency with contiguous areas to the disappearance of the line of separation is a disadvantage, as the accommodation of the eye might tend to vary unless the boundary is sufficiently definite; such variation would tend to prejudice the finest discriminations of contrast. In order to avoid this effect, means are sometimes provided to make the setting depend on a recognition of contrast in two parts of the field. (See the description of the Lummer-Brodhun prism.) Prejudicial effects due to visual fatigue may be avoided in a similar way by the use of the contrast principle.

The action of fatigue may be illustrated by a simple consideration of a strong brightness contrast between two contiguous rectangles A and B, where A is the brighter. The eye which observes this contrast grows more fatigued in the retinal area corresponding to the image of A, and hence when the physical intensities are equalized in the ordinary procedure the side A will apparently be less bright. If, however, B appears between two patches of A, the eye can be alternately directed to the two boundaries during equalization, and since any contrast step will be in opposite directions at the two sides of B, fatigue effects will be eliminated.

Size of the Matching Field.—There seems to be no doubt that the retino-cerebral visual receiving apparatus functions best under conditions in which the whole retina is working at about the same general level of brightness. Abney has described the effect when a large square patch of light of diameter subtending say 2° is contrasted with a small patch subtending say 20°, both at an equal low level of brightness; the small patch *appears* relatively darker, and this effect has been verified by many other observers. An important deduction is that the contrasting patches in a photometric field should be equal in area.

Experiments have been made on the threshold of vision—the minimum brightness visually perceptible—and its variation with the size of the stimulus source. It is found that up to an angular subtense of about 1° in the visual field the threshold brightness is nearly inversely proportional to the square of the linear dimensions of the stimulus, i.e. to the total amount of light received by the eye. Above 1° the threshold brightness is inversely proportional to the linear dimensions of the source (if this be a round or square patch), and this law holds up to an angular subtense of at least 4° or 5°. A similar law holds for the minimum perceptible brightness contrast between two equal rectangular patches whose size is varied. An object of low visibility in the centre of the visual field may be seen considerably more clearly by illuminating the peripheral portions of the retina. The present writer has also shown that a considerable improvement in the accuracy of photometric matching can be obtained by increasing the subtense of the total illuminated field from say 2° up to 30° or more. The actual patches to be matched may still remain comparatively small, while the peripheral portions of the retina are illuminated by an auxiliary field; but the great thing is to secure an approximately equal stimulation of the outer regions of the retina, and this may be done in any convenient way.

Brightness Level for Maximum Contrast Sensitiveness.—At very low illuminations it is clear that small percentage differences of brightness between two patches must tend to become invisible, especially if the patches themselves can barely be seen. On the other hand slight contrasts will be equally difficult to perceive when the general intensity of light becomes too great for comfortable vision. Between these extremes there lies a range of brightness levels at which the eye will be more or less sensitive to contrast. The most complete investigation of the contrast sensibility of the retina has been carried out by König and Brodhun, and their work has been verified by Blanchard and Nutting. The contrast sensibility is expressed as

$$\frac{dB}{B + dB},$$

where $B + dB$ is the stimulus which can just be distinguished from B. At the threshold the value of the sensibility will evidently be unity.

The table on p. 200 is reproduced in full as it is of importance in photometry and spectrophotometry. It gives results for spectral lights of a number of wave-lengths as well as for white lights. The estimated intensities of all the coloured lights and white light were equal at a high intensity, and the lower " intensities " were produced by reduction from the equal high intensities by a standard photometric method. See p. 202.

Nutting states that, according to his results, König's unit of intensity was 0·004 millilamberts. This unit of brightness, the millilambert, is the brightness of a perfectly diffusing surface which radiates or reflects 0·001 lumen per square centimetre. A perfectly reflecting and diffusing surface illuminated with an intensity of 1 foot-candle would possess a surface brightness of 1·076 millilamberts. The lambert, i.e. 1000 millilamberts, is thus a fairly large unit of

CONTRAST SENSIBILITY

Results of König and Brodhun

Wave-length.	·670.	·605.	·575.	·505	·470	·430.	White.
Field-Brightness.*							
1,000,000							0·0358
500,000							0·0273
200,000		0·0425					0·0267
100,000		0·0241	0·0325				0·0195
50,000	0·0210	0·0255	0·0260				0·0173
20,000	0·0160	0·0183	0·0205	0·0195			0·0175
10,000	0·0156	0·0163	0·0179	0·0181			0·0176
5,000	0·0176	0·0158	0·0166	0·0160			0·0179
2,000	0·0165	0·0180	0·0180	0·0175	0·0180		0·0181
1,000	0·0169	0·0198	0·0185	0·0184	0·0167	0·0178	0·0178
500	0·0202	0·0214	0·0180	0·0194	0·0184	0·0214	0·0192
200	0·0220	0·0225	0·0225	0·0220	0·0215	0·0245	0·0222
100	0·0292	0·0278	0·0269	0·0244	0·0225	0·0246	0·0298
50	0·0376	0·0378	0·0320	0·0252	0·0250	0·0272	0·0324
20	0·0445	0·0460	0·0383	0·0295	0·0320	0·0345	0·0396
10	0·0655	0·0610	0·0582	0·0362	0·0372	0·0396	0·0477
5	0·0918	0·103	0·0888	0·0488	0·0464	0·0494	0·0593
2	0·1710	0·167	0·136	0·0655	0·0715	0·0600	0·0939
1	0·258	0·212	0·170	0·0804	0·0881	0·0740	0·123
0·5	0·376	0·276	0·208	0·0910	0·096	0·0966	
0·2		0·332	0·268	0·110	0·127	0·116	
0·1			0·396	0·133	0·138	0·137	
0·05				0·183	0·185	0·154	
0·02				0·251	0·209	0·223	
0·01				0·271	0·289	0·249	
0·005				0·325	0·300	0·312	
0·002						0·369	
Threshold ..	0·060	0·0056	0·0029	0·00017	0·00012	0·00012	·000072

* Nutting states that the unit of brightness in this table corresponds to ·004 millilamberts. An illumination of 1 foot-candle on a perfectly reflecting and diffusing surface would give a surface brightness of 1·0761 millilamberts.

brightness, and is somewhere near that of a candle flame or of outdoor objects illuminated by strong sunlight. There are now quite a number of brightness photometers on the market, of which the British ones usually give results in " foot-candles ". A standard diffusing surface goes with the instrument, and when the brightness in foot-candles of any object is measured (as x say), the results indicate that " the brightness which the standard diffusing surface would possess, if illuminated by x foot-candles, is equal to the present brightness of the object ". If the reflection coefficient of the standard surface is r, a perfectly reflecting surface illuminated by xr foot-candles will be equal in brightness to the standard diffusing surface. Hence the surface brightness of the object in millilamberts $= 1{\cdot}076\ xr$. This assumes the standard surface to be perfectly diffusing, which will usually be sufficiently near the truth.

With the aid of a brightness photometer it is not at all difficult to make an estimate of the brightness of a photometer field, and this is very useful in securing the most suitable photometric conditions.

For a somewhat fuller discussion of the table, reference may be made to the writer's *Colour and Methods of Colour Reproduction* (Blackie).

Purkinje Phenomenon.—In spite of the essential difference in the sensations given by lights of different colours, it is possible to compare and match their brightness or luminosity. After a little practice it is easy to vary the brightness of a red patch so as to match that of a white, and to repeat the operation with considerable consistency in the result. We can thus take a whole series of bright colours, red, yellow, green, and blue, say, and vary their relative brightnesses till they are all equal. If the amount of light reaching the eye is then considerably reduced by some standard photometric device in exactly the same ratio for each patch, so that all are of a much lower brightness, it will be found that the apparent brightness of the patches rises from the red, which appears darkest, to the blue, which appears brightest.

Weber's law* as applied to sensations of luminosity may be stated as follows:

$$\frac{dB}{B} = \text{constant},$$

where dB is the least perceptible increase in the brightness of one of two contrasting patches of original brightness B. The foregoing table of König and Brodhun shows, however, that the fraction (often called the Fechner fraction) does not remain constant, but rises considerably for any colour as low levels of brightness are approached. Generally, however, the fraction decreases with shortening wavelength, for owing to the Purkinje effect the blue at a nominal brightness of 10 units, say, will *appear* brighter than the red of the same nominal brightness. (Remember that the lower brightnesses were produced by proportional reduction from equal high brightnesses.) The value of the Fechner fraction is thus considered to depend on the apparent *sensation* rather than on the numerical value of the stimulus; and Nutting states that, if the table had been corrected for the Purkinje effect, the values for the different hues would be sensibly equal and equal to the value for white at the same brightness level.

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* Sometimes known as Fechner's law. Also see footnote, p. 209.

Possible Accuracy of Photometric Settings.—Reference to the table shows that the least uncertainty of a single photometric setting is not much less than 2 per cent—at least under the conditions of experiment adopted in the investigation. The mean of several settings may, of course, be taken to lie much nearer to the truth. As was previously indicated, the present writer finds that if the peripheral regions of the retina are suitably illuminated, the probable error of a single setting may be considerably reduced.

Experimental Methods of Varying the Brightness of Light in Definite Numerical Proportions.*—A few notes on methods of varying the brightness of light will be of service in discussing particular instruments, but it will first be well to revise a few definitions and conceptions in regard to the meaning of " brightness ". One conception of the brightness of a surface is related to the quantity of light falling on unit area of the retinal image. Other things being constant, the amount of light falling on unit area of the retinal image will be proportional to the amount of energy radiated by unit area of the object. Hence the measurement of " brightness " is really a measure of " stimulation ". To vary the brightness of a diffusively reflecting screen in a known numerical way, it suffices so to vary the quantity of light falling on it (i.e. the " intensity of illumination "), provided that all the geometrical conditions of the direction of incidence of the light and the relative positions of screen and eye remain absolutely constant. Needless to say, the energy distribution in the spectrum of the light must also remain constant.

1. *Distance Variation.*—Since the amount of light remains constant for a tube whose walls are made up of confocal rays of light, it may be easily deduced that the intensity of illumination of a screen placed normal to the direction of a very small source will be inversely proportional to the square of its distance.

By means of lenses the image of a small source may be projected to any required position; the variation of intensity will afterwards take place with regard to the new distance of the image. Thus if the image is infinitely distant (" parallel " light) the intensity of illumination will remain constant in the parallel beam. In practice, however, the beam will usually depart from parallelism owing to the finite dimensions of the source. The use of a concave lens will often be useful to render a beam more divergent and thus of lesser intensity in a given cross-section.

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* A formal presentation of the theory of photometry will be found in Drude's *Theory of Optics*, Chapter IV.

2. *Variation of Angle.*—It is easily deduced geometrically that the intensity of illumination of a small flat screen is proportional to the cosine of the angle made by its normal with the direction of the incident light.*

" Lambert's law " states that unit area of a perfectly diffusing surface radiates or reflects with a power proportional to the cosine of the angle made by its normal with the direction of radiation or reflection. (N.B. The *power* is the quantity of light sent into unit solid angle in the specified direction.) Thus a perfectly diffusing surface will always appear of the same brightness from whatever direction it is illuminated.

The foregoing statements indicate two ways in which the intensity of illumination on a plane surface may be varied (i.e. variation of the distance of the source of light or its image, and alteration of the angle of incidence of the radiation), but in each case the effect of brightness to the eye depends entirely on the reflective properties of the screen which is illuminated. There are comparatively few surfaces which can be taken as perfectly diffusing within the meaning of Lambert's law. Consequently, should this principle be employed in an instrument, the action always needs independent calibration.

3. *Variations of Aperture.*—If an extended, uniform, diffusing source of light be illuminating a screen, it is clear that, when the dimensions of the source are small in comparison with its distance from the screen, the illumination on the screen will be proportional to the area of the source which is exposed. A somewhat similar case is encountered when a lens system forms a real image of a small uniform luminous source, as the posterior aperture of the optical system may be looked upon as a uniformly radiating surface, and indeed will appear as such to an eye placed in the position of the image and focused on the aperture.

On the supposition that the source is axial and diffuses light in accordance with the cosine law, it may readily be shown that the amount of light falling on unit area of the image will be proportional to $\sin^2 U'$, where U' is the maximum angle made by the image forming rays with the axis (fig. 132). The intensity of illumination of an image may thus be readily varied, and this device is of common use in photography. An eye placed axially so as to view the image may not, however, perceive any variations of brightness when the aperture is varied, because all the light entering the eye may usually be considered to originate from the central portions of the lens, and if these are not obscured the peripheral regions may be screened off without apparent effect.

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* Mr. A. P. Trotter states that this is the original Lambert's law, and that the law which is usually known by that name is not due to Lambert.

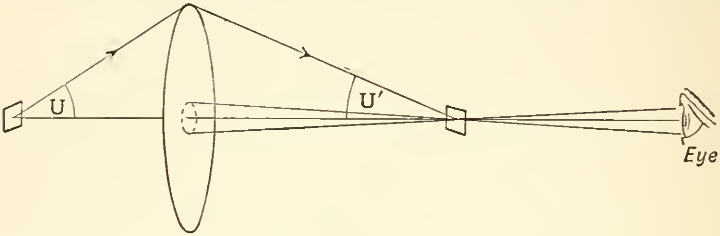

Fig. 132

If a translucent diffusing screen be placed in the plane of the image, the apparent brightness of the image as viewed from behind will *then* be proportional to $\sin^2 U'$, provided that the screen perfectly diffuses the incident light.

It will be realized from the foregoing sections that the properties of materials play a great part in photometry. They have at present been by no means sufficiently fully studied.

It must be remembered that the basis of theoretical photometry is the assumption previously stated, i.e. the amount of light remains constant inside a tube the walls of which are composed of confocal rays of light. When dealing with the passage of light through small apertures, however, the postulate breaks down, as a considerable amount of energy is diffracted away from the geometrical ray paths, and the action increases the narrower the aperture. Diffraction effects have therefore to be guarded against when it is found that the action of an instrument depends on the employment of a variable aperture over an image-forming lens.

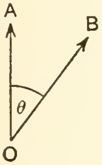

Fig. 133

4. *Polarization Methods.*—Of all the means of varying the intensity of light, the employment of polarizing prisms is one of the most frequently used in instruments. A suitably constructed prism *entirely* polarizes the part of a beam of light which it transmits in certain directions. Imagine that, referring to fig. 133, a beam of

polarized light, with vibrations of amplitude A, is coming from the point O towards the eye. Suppose that the prism which has produced the polarization transmits only such vibrations as are parallel to OA. If a second prism, which only transmits vibrations parallel to OB, is then interposed, the amplitude is theoretically reduced to $A_1 \cos\theta$, where θ is the angle between OA and OB. The employment of two such prisms therefore permits of the quantitative variation of the intensity of the transmitted light from zero up to a maximum, A_1; the intensity is found, confirming the theory, to be proportional to $\cos^2\theta$.* Angular displacements are easy to read to a considerable accuracy, and the system is a convenient one.

The Nicol Prism.†—Fig. 134 (*a* and *b*) shows two views of the Nicol prism. The first illustrates the method of cutting the well-known rhomb-shaped prism of Iceland spar, and the second is a section made by the plane through the shorter diagonals of the end faces. The rhomb of spar before cutting is of such a shape that the long edge AB is about one and a half times the length of the shorter diagonal, BC, of the end face. The rhomb is cut with a fine saw; the cut faces are polished, and afterwards cemented together with Canada balsam or (very often) linseed oil. In some cases the Nicol may be trimmed to make the long edges of the prism parallel to the central ray of the useful field, or (for another purpose) by cutting faces perpendicular to the length of the prism through the points A and C (fig. 134 *b*).

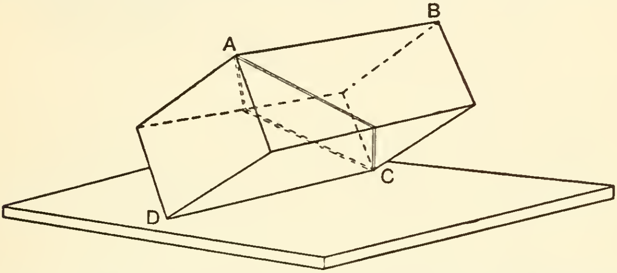

Fig. 134*a*.—Method of cutting Rhomb of Iceland Spar to make a Nicol Prism

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* It has been sometimes questioned whether it is in every case sufficiently accurate to assume the truth of this law. It is most likely to be misleading near the zero point, when $\cos\theta = 0$. See p. 211. A Nicol never transmits more than half the light from an unpolarized beam.

† Many useful details on polarizing prisms will be found by reference to papers by S. P. Thompson: (1) *Proc. Optical Convention*, 1905 (The Optical Society, London); (2) *Proc. Phys. Soc.*, Lond., Vol. II, p. 185.

It may be considered that, on entering the prism, a ray splits up into ordinary and extraordinary rays. Let the incident ray swing through various angles of incidence, starting from the condition when the angle of incidence of the light on the interior film of balsam is smallest. At first both the rays are transmitted through the film, but when the angle of incidence thereon of the ordinary ray increases to 67° 53′ it is totally reflected; there is a slight interference effect and the red rays are the last to be totally reflected. There is a considerable angular range in which the extraordinary ray continues to be transmitted by the film while the ordinary ray is completely reflected, but when the angle of incidence approaches 82° 32′ the limit of transmission of the extraordinary ray is reached, owing to the increasing index of refraction now valid for the extraordinary ray as the direction of the optic axis is approached.

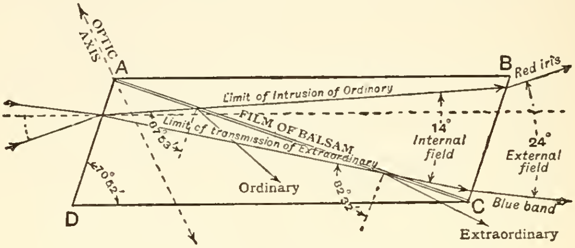


Fig. 134*b*.—The Section Plane as in a Nicol Prism

There are many ways of cutting up Iceland spar to produce polarizing prisms; the main art in doing so consists in producing the arrangement which will give the best angular field. Fig. 135 shows a diagram of the "Glan-Thompson" prism, in which the faces are parallel to the optic axis and to the visual axis. This prism, when cemented with Canada balsam, gives a beam of polarized light of angle 27°.

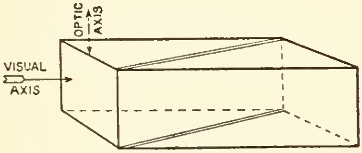
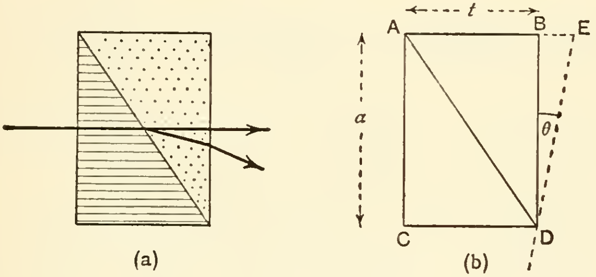


Fig. 135.—Glan-Thompson Prism

Prisms with end faces which are not normal to the transmitted light produce a certain amount of elliptical polarization in the transmitted light. This is avoided by normal end faces as in the Glan-Thompson prism. Furthermore the normal end faces in a prism are an advantage from an optical point of view because, if a convergent or divergent beam is passed through it, rotation of the prism does not displace the image. This is a serious drawback to the use of the ordinary Nicol prism in optical instruments, so that Glan-Thompson prisms are usually employed in photometric instruments, especially in rotating parts. The same action takes place when light is transmitted through oblique refracting surfaces in any instrument. If such a transmission takes place between the polarizer and analyser of, say, a polarizing spectrophotometer, special precautions such as described in connection with the Hüfner instrument must be taken.

Double-image Prisms.—There are two useful types of prism which will split a beam of ordinary light into two diverging portions polarized in perpendicular directions. They may be made of quartz or of Iceland spar accordingly as small or large angular separations respectively are required.

Fig. 136 shows the Rochon prism, and the direction of the optic axis in each portion is indicated by shading. In the right-hand part it is perpendicular to the diagram. The ordinary ray travels along the axis in the first part and perpendicular to the axis in the second part. The extraordinary ray has a velocity equal to the ordinary in the first part along the optic axis, but changes velocity on entering the second portion. Hence there is a deviation of the extraordinary ray.



(a) (b)

Fig. 136.—Rochon Prism and Theory

In order to calculate the deviation approximately, a simple treatment on the theory of optical path will be useful. Let ABDC be a prism with end faces accurately normal to the edges AB and CD (fig. 136 (*b*)). A wave-front coincident with AC would after refraction through the prism be parallel to BD just after emergence if the prism were all of the same material and homogeneous.

Consider, however, the extraordinary ray in the Rochon prism. From A to B the refractive index concerned is the refractive index for the extraordinary ray when travelling perpendicular to the optic axis, i.e. the " extraordinary refractive index ". From C to D the apposite refractive index is the ordinary refractive index, since all this path is parallel to the optic axis of the crystal.

In general the optical paths from A to B and C to D will not be equal, and the wave-front of the extraordinary ray on emergence from the crystal will pass through E and D, say. E lies on AB produced.

If the deviations are small the optical path ABE must equal CD.* Hence, calling the extraordinary and ordinary refractive indices μ_e and μ_o respectively, we write:

$$\mu_e t + a\theta = \mu_o t,$$

or

$$\theta = \overline{\mu_o - \mu_e} \cdot \frac{t}{a}.$$

This equation will give a fair approximation to the deviation up to fairly large angles (10° or so) and is very useful for calculating the thickness and angles of prisms required to give a prescribed approximate deviation. The calculation is otherwise somewhat troublesome.

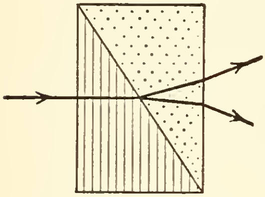


Fig. 137.—Wollaston Prism

Sometimes the Rochon prisms are made with the first component of glass; they may be calculated in a very similar way.

Fig. 137 shows the Wollaston prism, in which both ordinary and extraordinary rays are oppositely deviated, each to the same amount as the extraordinary ray in the Rochon prism of the same dimensions. The light travels perpendicular to the axis in each part; in the first part the optic axis is parallel to the front face and to the plane of the diagram, while in the second part the axis is perpendicular to the plane of the diagram.

In polarization photometry the eye usually views the contrasting parts of a photometric field through an analyzing prism. In some cases the light in one half only of the field may be plane polarized, when its intensity will vary, as above, on rotation of the analyser. The intensity of the other half remains constant. When the match is effected we shall have

$$I_0 \cos^2\theta = I,$$

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* By Fermat's theorem. See p. 159, also Schuster's *Theory of Optics*, p. 43.

where I_0 is the maximum brightness of the polarized portion and I the steady brightness of the unpolarized portion; θ is the angle of rotation from the position of maximum brightness. Hence if θ is measured the ratio of I_0 to I may be calculated.

Sensitiveness of the Setting.—The brightness B of the variable portion is given by

$$B = I_0 \cos^2\theta.$$

Hence $$\frac{dB}{d\theta} = -I_0 \sin 2\theta.$$

Supposing that a small change $d\theta$ were made from the angle θ at which a match had been made, the visibility of the contrast, according to Weber's law, will depend on the ratio of the change of brightness dB to the first brightness B.

$$\frac{dB}{B} = -\frac{I_0 \sin 2\theta}{I_0 \cos^2\theta} d\theta = -2 \tan\theta d\theta.$$

When $\delta\theta$ is the least angle which will cause a perceptible contrast we shall have $\delta\theta = -\frac{\delta B}{B} \cdot \frac{1}{2 \tan\theta}$, where $\frac{\delta B}{B}$ is the visual Fechner fraction. We notice that $\tan\theta$ reaches an infinite value when θ is 90°, hence $\delta\theta$ would approximate to zero if $\frac{\delta B}{B}$ were finite; but this would entail a match made at zero brightness, which would be valueless. Owing to the diminution of general brightness of the field when matches are made at increasing values of θ, it will be found that the visual Fechner fraction, as thus defined,* increases more rapidly than $\tan\theta$ above, and no sensitiveness will be gained by increasing θ any further. Provided that sufficient light is available, it is an advantage to arrange matters so that the match may be made when θ is fairly large, for high sensitiveness in the settings is a decided advantage in photometry, since the liability to personal and mechanical errors is diminished. The law of variation of brightness is, however, most uncertain near the extinction point, owing to the "Landolt band" effect (see below) and the possible presence of stray light.

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* Note the difference between $\frac{\delta B}{B}$ and $\frac{\delta B}{B + \delta B}$. The latter fraction was given in the table on p. 200, and only becomes unity at the threshold, whereas $\frac{\delta B}{B}$ becomes infinite.

In some cases the light in the two halves of the photometric field is polarized in perpendicular directions. In fig. 138 (*b*), let OA represent in direction and magnitude the amplitude of the vibrations in the left-hand part of the field and let OB similarly represent those in the right-hand portion. The analysing Nicol is then turned so as to transmit vibrations parallel to some direction OP, when the resultant intensity of the two halves of the field is equalized. Then

$$\mathrm{OA}\cos\theta = \mathrm{OB}\sin\theta,$$

and $$\frac{\mathrm{OA}}{\mathrm{OB}} = \tan\theta,$$

whence $$\frac{\mathrm{I}_0}{\mathrm{I}_0'} = \tan^2\theta,$$

where I_0 and I_0' are the primary intensities of the light in the left- and right-hand parts of the field.

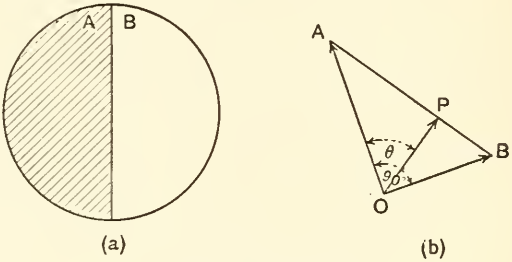

Fig. 138

The sensitiveness of the settings may be briefly discussed. Let the intensity of the left-hand part of the field as seen through the analyser be B_1, and the intensity of the right be B_2; then

$$\mathrm{B}_1 = \mathrm{I}_0\cos^2\theta, \quad \text{and} \quad \mathrm{B}_2 = \mathrm{I}_0'\sin^2\theta.$$

Hence, taking differentials,

$$d\mathrm{B}_1 = -\,\mathrm{I}_0\sin 2\theta d\theta,$$

and $$d\mathrm{B}_2 = \mathrm{I}_0'\sin 2\theta d\theta.$$

If the fields were equalized before the analyser was turned through an angle $d\theta$, the difference in relative brightness $d\mathrm{B}$ is now given by

$$d\mathrm{B} = d\mathrm{B}_2 - d\mathrm{B}_1 = (\mathrm{I}_0 + \mathrm{I}_0')\sin 2\theta d\theta.$$

As above, the visibility of the contrast is best considered by the fraction $\frac{dB}{B}$. We obtain

$$\frac{dB}{B_1} = \frac{I_0 + I_0'}{I_0} 2 \tan\theta d\theta = \frac{4d\theta}{\sin 2\theta}.$$

Very much the same conclusions will therefore be reached in the present case. The sensitiveness will be greatest when θ is near 0° or 90°, and the scale will be most open when the two parts of the field are nearly equal so that θ is approximately 45°. As in the previous case the equation may be written

$$\delta\theta = \frac{\sin 2\theta}{4} \times \text{(Fechner fraction)},$$

where $\delta\theta$ is the least rotation required to produce a perceptible contrast.

Considerations of the foregoing kind are applicable to photometric settings in polarimetry, &c. (p. 246).

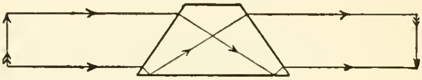

Fig. 139

Landolt's Band.—In the majority of cases the light passing through a polarizing prism is not strictly parallel, and in consequence of the differing obliquities of the rays incident at the various angles, the direction of vibration of the polarized light varies in different parts of the field to a very slight extent. Hence when a *very intense* source of light is examined with crossed Nicols the whole field does not become uniformly black, but the field is crossed by a black band which moves from one side to the other with a very small rotation of the analyser.[*]

5. *Rotating Sectored Discs.*—Rotating sectored discs may be used to vary the apparent visual intensity of a beam of light passing through them, provided that the speed is sufficiently great to eliminate the appearance of flicker. Talbot, Plateau, Abney, Hyde, and others have conducted experimental researches which have proved that the apparent visual intensity of the transmitted light is proportional to the angular aperture of the sector. Especially convenient are the adjustable sectors of which the angle can be varied during rotation. The use of a suitable erecting prism (fig. 139), rotated about an axis parallel to its base with half the speed of the sector, permits sector readings to be taken when the apparatus is in motion. It is

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[*] Lippich, *Sitz. Wein. Akad.*, 1882, LXXXV, p. 268.

advisable, however, not to reduce the angular aperture of the opening below 10°, otherwise a loss in accuracy is to be expected.

6. *Light Filters and Wedges.*—It frequently chances that none of the foregoing methods are so convenient as one of the various types of photometric " wedge ". This may be a photographic wedge, a wedge of gelatine having lampblack in suspension and cast in a suitable mould,* or one of glass (fig. 140). In the last case the wedge of " neutral " glass is usually cemented to one of colourless glass, oppositely placed in order to eliminate deviation.

The intensity of the light passing through a wedge depends on the thickness of the absorbing medium traversed. The equation

$$I = I_0 e^{-\alpha t}(1 - \beta)^2$$

represents the intensity of the transmitted light for a glass wedge (sufficiently nearly), where I_0 is the intensity of the incident light, e is the base of natural logarithms, t is the thickness of absorbing medium traversed, α is the absorption coefficient, and β is the reflection coefficient for a glass-air reflection. If the wedge is uniform, the thickness will be proportional to the distance (from the apex of the wedge) of the portion traversed by the light, and the intensity of the transmitted light is thus a logarithmic function of a measureable displacement. In practice the wedges cannot be produced sufficiently uniform to render unnecessary the precaution of a careful calibration.

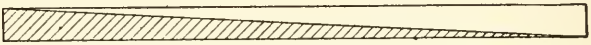

Fig. 140.—Photometric Wedge of Glass

The foregoing remarks should be sufficient to draw attention to many points which often escape notice in the discussion of photometric instruments, and consideration can now be given to the actual construction. Attention should be directed, however, to the section dealing with experimental methods in making photometric settings (p. 237).

Illumination and Commercial Photometry.—It is not within the scope of the present book to deal at any length with the instruments and methods of illuminating engineering photometry. The photometer bench differs from the optical bench only inasmuch as an easy motion of the carriages is necessary; length measurements need not generally be taken to such a high order of accuracy, although a greater "over-all" length is required. The fittings generally move on rollers, and they may be shifted by means of cords or rods while the observer looks into the centrally supported Lummer-Brodhun photometer head. The *sine qua non* is that no light, except that which is derived from the sources to be compared, shall fall on the standard surfaces of the photometer head. This condition is usually secured by the liberal use of black painted diaphragms and curtains of black velvet as suggested in fig. 141. The screen S of the "head" may be removed entirely, and a convenient test for stray light is thus to remove it and look through the windows of the fitting towards the lamps in each direction to see if any stray light is visible.

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* Goldberg's wedge and its construction was described at the Photographic Conference, Brussels, 1910. See also *Brit. Jour. Phot.*, 1910, Vol. LVII, p. 648; *Phot. Jour.*, 1911, p. 414; *Proc. Faraday Society*, November, 1923; Conference on Physical Chemistry of Photographic Process. It is made in Great Britain by Messrs. Ilford, Ltd.

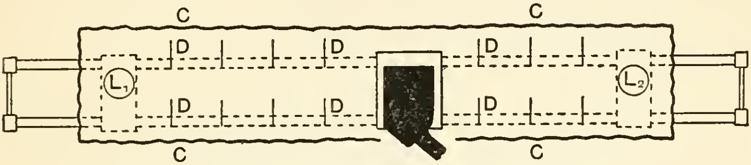
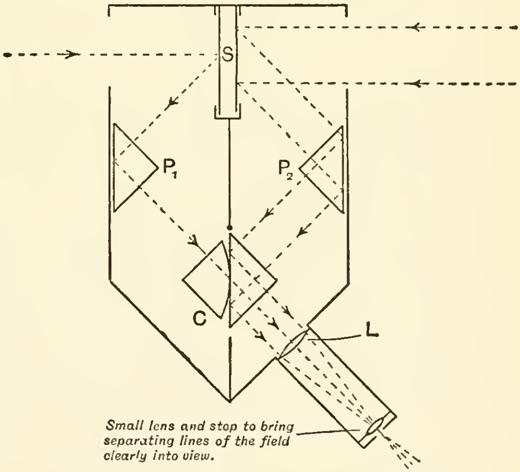


Fig. 141.—Photometric Bench with Diaphragms and Curtains

L_1, L_2, Lamps under comparison. C, Curtains (black velvet). D, Diaphragms.



Small lens and stop to bring separating lines of the field clearly into view.

Fig. 142.—Lummer-Brodhun Photometer Head

Lummer-Brodhun Photometer Head.—A few remarks on the optical principles of the Lummer-Brodhun head may be of service. The general arrangement is shown in fig. 142. S is a reversible screen on each side of which is a fine white diffusing surface. It may be removed entirely if desired. The whole head is held on the photometric bench so that this screen is normal to the light from the lamps under comparison. Light is diffused from each side of S and is reflected by the mirrors or prisms P_1, P_2 on the interior walls of the box to the compound prism C. The mirrors are often replaced by totally reflecting prisms.

A few forms of the compound prism, the earliest type of which was invented by Swan in 1859, are shown in fig. 143. In No. 1 two right-angled prisms are taken. These if cemented with their hypotenuse faces together would form a cube. The outer portions of this face of one prism are removed with a shallow spherical grinding tool, thus leaving a central circular portion still polished. This portion is then applied to the other prism with such a pressure that the clean faces adhere in " optical contact ". When this is the case the film of air between the surfaces is small compared with the wave-length of light, and no reflection takes place at the boundary.

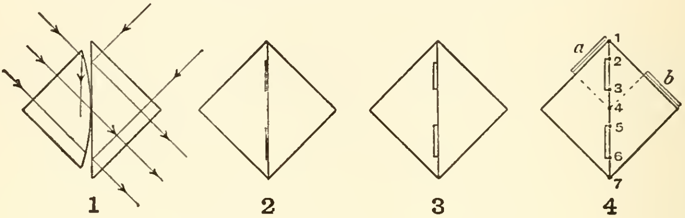


Fig. 143.—Forms of Photometer Prisms

The central portions of each beam thus pass freely through the contact, but the outer portions of that beam incident from the right will be totally reflected at the glass-air surface, and these will travel to the lens L (fig. 142) together with the central part of the beam from the left. A small circular telecentric stop about 4 to 5 mm. in diameter is placed at the principal focus of the lens L; this stop fulfils an important function. It permits only those rays which were parallel to the axis of the lens before refraction to pass through the stop to the eye, and the direction of this axis is at 45° to the contact face. This ensures, firstly, that none of the light scattered from the " ground " outer surface of the left-hand prism shall reach the eye, for it cannot be refracted into the right-hand prism at an angle with the contact face greater than the critical angle, which is itself less than 45°; secondly, it ensures that all the light reaching the eye shall come from one face or the other of the diffusing screen S. The eye behind the stop is accommodated to view the contact surface of the prism, sometimes with the aid of a small lens shown in the figure adjacent to the stop.

The application of the other forms of the compound prism will readily be understood. In No. 2 certain portions of the contact face are silvered before cementing. The silvered surface acts as an efficient but not perfect reflector. It is also easy to arrange the shape of the portions of the field in any way required, but the edge of the silver is sometimes a little ragged.

In No. 3 certain portions of the left-hand prism are removed by sand-blasting instead of grinding.

No. 4 employs the " equal contrast " principle. The main prism is of similar construction to that in No. 3, but two thin plates of clear glass which obstruct a small quantity of light are placed outside the prism in the positions shown in the figure. As seen from the ordinary viewpoint, the parts 1–2 and 3–4 are illuminated by light from the left diminished by the outer plate; the part 2–3 is illuminated by undiminished light from the right. Again, the parts 4–5 and 6–7 of the separation surface are illuminated by undiminished light from the left, but the part 5–6 is illuminated by diminished light from the right. The effect is that a complete matching in the entire field is not obtained, but the setting is made when the contrast between the part 2–3 and the surrounding portions is equal to the contrast between 5–6 and its surrounding parts. The use of this device is somewhat less fatiguing to the eye than the " equality " type of setting, but it has not been generally found to give any great increase of accuracy. Comparatively recently Hyde and Cady have made experiments regarding the effect of variation of contrast in such a field, with the result that some improvement in the accuracy of settings seems possible when the correct contrast is employed. Photometers employing the equal contrast principle have by far the most widespread use for optical bench work.

For details regarding photometer heads depending on the " flicker " principle reference must be made to textbooks on " Photometry ".

Illumination Photometer, König-Martens Type.—The principle of the optical system to be described is embodied in several instruments, notably the König-Martens photometer, the spectrophotometer of the same name, and the Wanner type of optical pyrometer. Slightly different arrangements are illustrated in figs. 144 and 145. In fig. 144, *a* and *b* are two small circular apertures, from which light diverges; the beams are collimated by the lens L_1, and proceed to the Wollaston prism W (p. 208). Each beam is now divided by the prism into two parts, plane polarized in perpendicular directions and mutually slightly divergent, so that there are four beams in all. The light then traverses the combination of lens and biprism L_2P, and each beam is further subdivided by the biprism, so that eight images are produced in the focal plane of the lens L_2. Of all these it is arranged that two images, one of each aperture, fall in the aperture *c* in the stop S and their light passes into the eye. One image is produced by light from the top section of the biprism and is plane polarized in a definite direction, while the other image is produced by light from the lower section and its direction of polarization is perpendicular to that of the other. Both beams traverse the analysing prism N and the lens L_3, which brings the dividing edge of the biprism into focus for the eye placed behind the stop. The analyser can be rotated and its angular position is indicated by a divided circle and pointer or vernier. This produces the condition

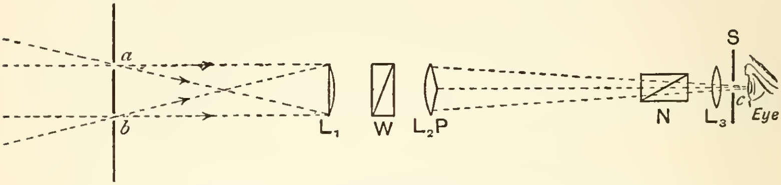


Fig. 144.—König-Martens Type of Photometer: to illustrate optical system

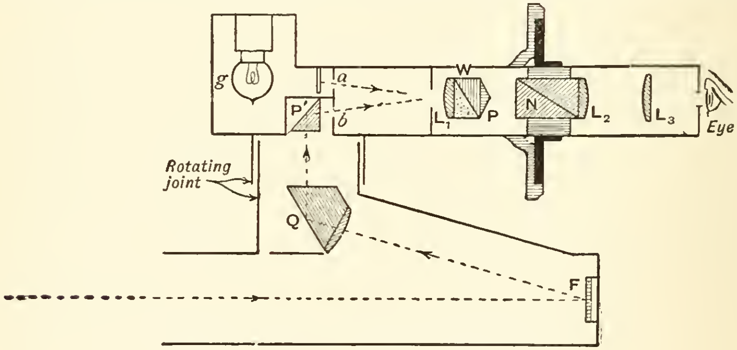


Fig. 145.—König-Martens Illumination Photometer

discussed on p. 210, in which the two parts of the photometric field are polarized in perpendicular directions and are viewed through an analysing Nicol.

It is convenient to illuminate the aperture *b* by means of an auxiliary constant source, usually a small electric lamp, and the light from this thus illuminates the portion of the field on the further side. The brightness of the other part of the field is proportional to the brightness of the surface, wherever it is, which sends light through the aperture *a* to the collimating lens L_1. The brightness of such a surface may therefore be compared with the standard furnished by the lamp, for which a battery, regulating resistance, and ammeter is provided, as well as a standard pentane lamp against which the standard may be checked. The luminous surface to which the instrument is directed must subtend a greater angle at the pin-hole than the diameter of the field, and in order that the whole semicircle may be uniformly illuminated the surface must be uniformly bright also.

The complete arrangement of the original Martens illumination photometer is shown in fig. 145. It will be noticed that the optical arrangement is somewhat different. The collimating lens and biprism are cemented to the double image prism, and a Ramsden eyepiece (lenses L_2 and L_3) is employed which brings the dividing line of the field into clear view. The front lens of the eyepiece is cemented to the analyser. The prisms P′ and Q secure the illumination of a part of the field by light reflected from the magnesia slab F, the intensity of illumination on which is to be tested. The tube containing the slab can be rotated with respect to the other tube about an axis perpendicular to both. The standard illumination is derived from the electric lamp *g*. The optical action of the instrument differs only slightly from that described in the previous paragraph.

It is to be noticed that of the light actually reaching the collimator lens from each source only one quarter at best would be transmitted by the final diaphragm. In addition to this the losses by absorption and reflection in the optical parts must be borne in mind. Probably not more than an eighth of the light reaches the eye, and in this respect the system is an extremely wasteful one.

Illumination Photometers.—In recent years the study of illumination has become of increasingly great importance, and a number of instruments on simpler and less expensive lines than the above type have come into widespread use. In the majority of types the illumination at a given point is determined by the measured brightness of a standard test surface placed in the required position; or alternatively this standard test surface may be an integral part of the instrument. In the *Sharp-Millar* and *Weber* types the brightness of the standard surface is contrasted, by the aid of a Lummer-Brodhun cube, with that of a screen inside the instrument; this screen is illuminated by a small electric lamp run under standard conditions, and it is arranged that the illumination may be varied by adjusting the distance of the lamp from the screen or vice versa. The distance of the lamp from the screen, as given by a scale, may thus be made to indicate " illumination ". In the Trotter photometer the standard test surface (matt white celluloid) is situated on the top of the box enclosing the instrument, and the observer views, through an aperture in the centre of the standard, a second plane white surface in the interior. The brightness of the latter is varied by altering the angle of incidence of the light from a small electric lamp which illuminates it, and which, as before, has to be run under standard conditions. For further discussion of these instruments, which are somewhat outside the scope of this book, see *Lighting and Photometry*, by J. W. T. Walsh (Methuen).

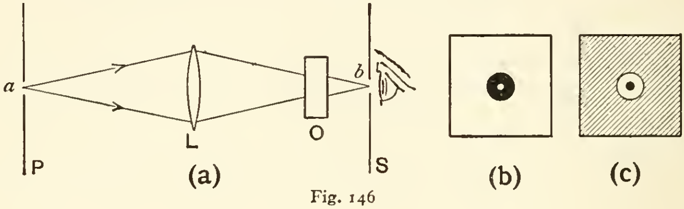


Fig. 146

The Effect of " Stops " or Diaphragms.—There are several general points in connection with the design of photometric instruments, especially those employing a somewhat complex optical system, to which attention may be directed. The first may be appreciated by reference to fig. 146.

A uniform beam of light diverges from an aperture *a* in a screen P to a lens L, which brings it to a focus in the plane of a screen S containing a small aperture *b*. On its way to the focus the light has to traverse some other optical parts which may or may not be movable. An eye placed behind *b* sees the lens L, if it is free from aberration, uniformly illuminated, provided that the light forming the image passes through the aperture in the screen (fig. 146 (*b*)).

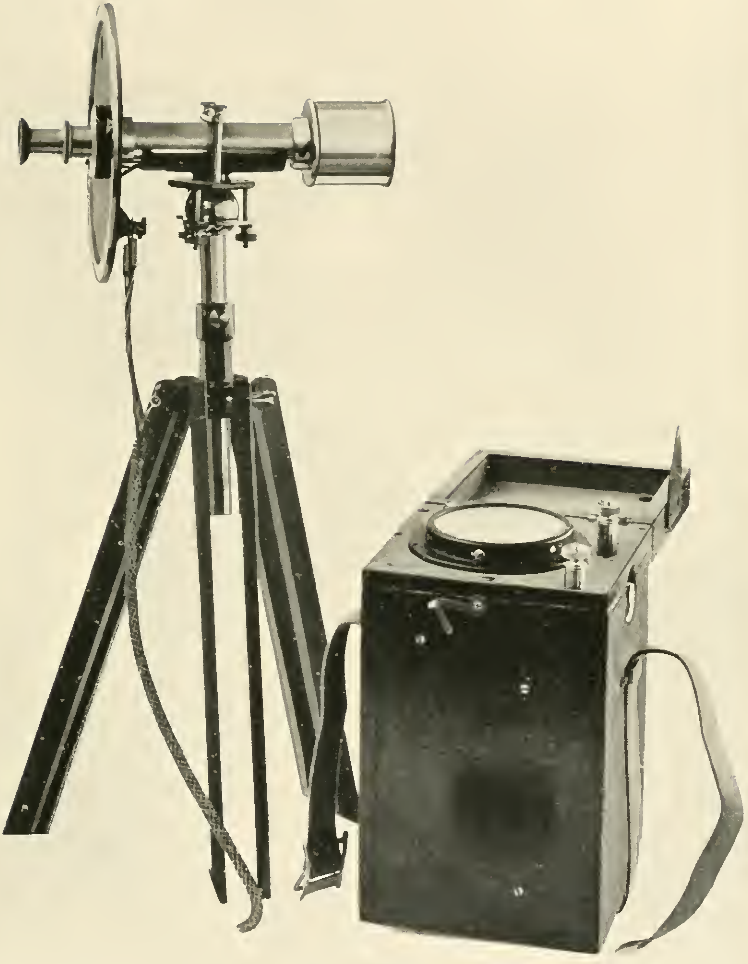


Fig. 147.—Cambridge Optical Pyrometer
(With case for battery, &c., for the standard comparison lamp)

If any shift of the image is liable to occur through motion of the optical parts one of two conditions must be satisfied in order that the intensity (to the eye) of the field may not appear to alter. Either the aperture must be large enough to allow the image always to be formed within it, no matter what shift takes place, or the image must be so much larger than the aperture that it always covers it (fig. 146 (*c*)). In this case also the image must be uniformly bright. If a constant area of the image is not contributing light to the eye, serious inaccuracy may result. This consideration is of peculiar importance in such a photometer as one of the type just described.

In one case recently encountered by the writer a "pointolite" lamp was employed as a source of light for a photometer, and the exit pupil of the instrument really consisted of two images of the glowing ball of tungsten with part of the rod near it. The light forming these images in the plane of the lens of the eye formed, when reaching the retina, the wider images of the two parts of the field of view. These images of the source, however, were not coincident and were liable to considerable relative motion when the analysing prism of the photometer was rotated. This was sufficient in some positions to bring the image of a portion of the source outside the pupil of the eye, which caused a consequent dimming of the corresponding part of the photometric field.

Stray Light.—Stray light is one of the most serious bugbears to the optical designer. Every pair of glass surfaces sends forward with the main beam a certain proportion of light which may form independent and surprisingly bright images in unexpected places. It is especially troublesome in photometric instruments, and, as will be shown when describing the König-Martens spectrophotometer, special precautions have often to be taken against it. From this point of view any increase in the optical complexity of a photometer system is a bad feature.

Stray light is especially troublesome in certain cases which may be considered by supposing the screen P (fig. 146) to be a radiating or reflecting surface with a luminosity varying greatly from place to place. On the lines of geometrical theory, the apparent brightness of the field will be proportional to the light derived from the region of the screen which conjugates to the aperture *b* with respect to the lens L. If, however, this region happens to be of very low brightness while the surrounding regions of the screen are comparatively bright, it is probable that stray light from these parts, scattered from the optical surfaces of the system, will considerably enhance the apparent brightness of the field. This can be guarded against by screening off the unrequired parts of the luminous surface.

Optical Pyrometer, Wanner Type.*—The intrinsic brightness of a perfect radiator for any given wave-length of visible light depends on the temperature only. The action of the Wanner optical pyrometer, which has an optical system essentially similar to that of the König-Martens photometer described on p. 216, depends on this fact. A narrow range of wave-lengths is isolated by means of a filter of red glass placed immediately before the eyepiece diaphragm, and the measurement of intrinsic brightness is effected with reference to the standard brightness of the part of the field illuminated by the auxiliary electric lamp. In calibrating the instrument, readings are taken on a few standard radiating sources at known temperatures, and interpolation is made on the basis of Wein's radiation law, which holds sufficiently well for the shorter wave-lengths. The instrument, as made up by the Cambridge and Paul Instrument Company, is shown in fig. 147. By placing a dark filter in the path of the beam from the test source the instrument may be utilized for reading over a second and higher range of temperatures.

The general subject of pyrometry does not come within the scope of the present book; for a fuller discussion, reference may be made to Ezer Griffiths's book on *Methods of Measuring Temperature*. The general considerations relating to other photometers apply, however, equally well to the Wanner pyrometer. The lens in front of the instrument acts as a window for the exclusion of dust, and also serves to bring images of external objects near the sighting direction into sharp focus in the plane of the biprism. This is a convenience in adjusting the instrument for sighting into furnaces, as the instrument then functions optically as a telescope system so far as one half of the field of view is concerned.

Spectrophotometers.—For many purposes it is often desirable to determine the manner in which the relative intensities of certain radiations which give continuous visible spectra vary with the wave-length of the light. A relative variation of intensity with wave-length may be produced by the passage of the light through a medium manifesting selective absorption or reflection. The effect of the medium may be found by taking comparative measurements with and without its presence. Instruments suitable for the measurement are known as spectrophotometers, and five types will be described: (1) the König-Martens, (2) the Hüfner, (3) the Nutting,

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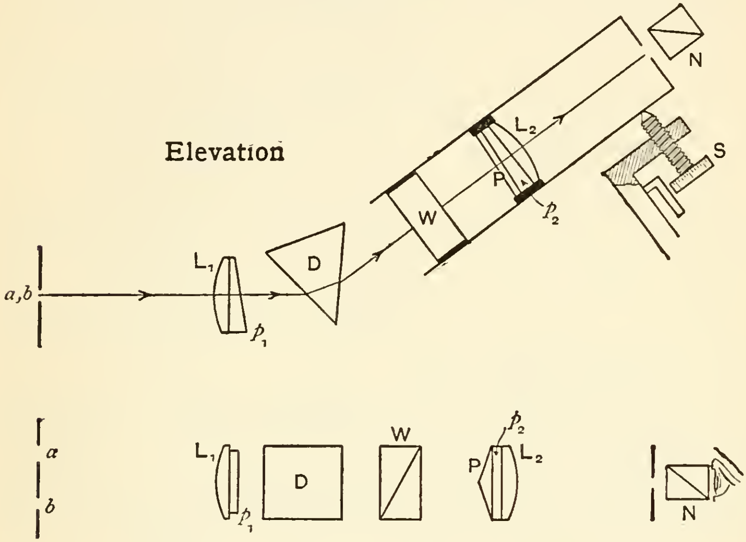
* Made by the Cambridge and Paul Scientific Instrument Co., Ltd.

(4) the Bellingham-Stanley, (5) Bull's non-polarizing spectrophotometer.*

König-Martens Spectrophotometer.†—The instrument embodies a photometer system similar to that described on p. 216, with the addition of a dispersing prism. This naturally produces its dispersion in a plane perpendicular to the plan section indicated in fig. 145, as is shown in fig. 148.

The right-hand arm of the instrument moves in a vertical plane about a horizontal axis passing through the 60° prism, and its position is controlled by a micrometer screw S which indicates the wavelength of the light illuminating the field. The lettering in the figure corresponds to that in the previous descriptions of the non-dispersing photometers, but attention must be drawn to a few special points. The dispersing prism D will easily be recognized. For monochromatic (spectrally pure) light the action will not differ at all from that of the simple photometer except for the upward deviation of


Elevation

Plan (not allowing for upward deviation)

Fig. 148.—König-Martens Spectrophotometer

the beam. The small prisms p_1 and p_2 serve the purpose of deviating out of the path of the main beam certain light which suffers double internal reflection at the surfaces of the optical parts. Apart from these special features, the action is similar to that of the other forms described. Kayser condemns this instrument on account of the waste of light.

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* Various other forms of the instrument have been devised, amongst which are those of Lummer and Brodhun, Brace, and Wild. These are described by Kayser in Vol. III of the *Handbuch der Spectroskopie*, but they are little used in this country.

† *Annalen d. Phys.*, Band XII, p. 984.

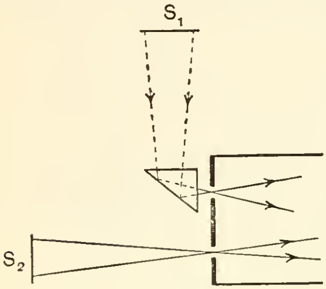

Fig. 149a

When it is necessary to compare the spectral distribution of the relative intensity of different sources of radiation, the arrangement suggested in fig. 149*a* may be adopted provided that the sources S_1 and S_2 are sufficiently extended and uniform, or the sources may be placed behind small discs of ground glass or opal glass; but if small sources are to be dealt with, lenses should be placed to throw images of them on the two apertures of the instrument. The aperture of each lens then functions as an extended source of light. In order to perform measurements of the absorption of light filters, it is desirable to arrange so that each aperture of the spectrophotometer is illuminated by the same source; this is conveniently ensured by the arrangement described by Martens and Grünbaum and shown in fig. 149*b*. S is the source of light just outside a slit *s* behind which a diffusing screen may be placed to equalize the distribution of light. L is a system of three lenses cemented together as indicated in the section; this system produces two real images of *s* which are brought to a focus on the apertures of the spectrophotometer by the two small Fresnel prisms f_1 and f_2. In this way the centre rays of the beams from the lenses 2 and 3 can be kept far enough apart to introduce the absorption tubes T_1 and T_2, which receive liquids for the measurement of absorption. Each part (2 and 3) of the composite lens then acts as an extended source, and the angle subtended at the apertures by these apparent "sources"

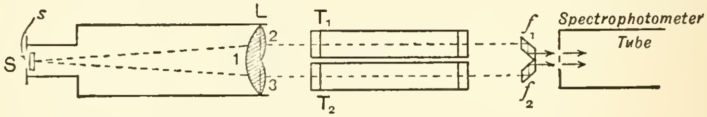

Fig. 149*b*

must be large enough to fill with light the respective halves of the photometric field when the parts are in correct adjustment. A pointolite lamp of the fixed focus type makes a suitable source of high intensity. For measurements on the absorption of a glass light filter it may simply be placed in the path of one of the beams.

The Hüfner Spectrophotometer,[*] which is made by the Optisches Institut, Hamburg, has many features which render it very valuable. In some points it resembles the Nutting instrument, as it provides a means for dividing the spectrum horizontally by optical parts placed in front of the slit of a spectrometer. One section is polarized and the test substance is placed in the other beam. The analysing prism, is, however, located in front of the object glass of the spectrometer telescope, where its rotation is measured with the aid of a divided circle and verniers. The spectrophotometer cannot therefore be added to a spectrometer as an entirely separate fitting.

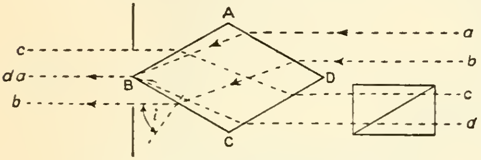

Fig. 150.—Hüfner's Spectrophotometer

The method of dividing the field is clearly shown in fig. 150. ABCD is a rhomb of glass, of which the refracting edge B is placed in juxtaposition and perpendicular to the slit of the spectrometer. The bundle of parallel rays *ab* incident from the right is seen to be lowered by the transmission through the parallel sided block, while the bundle *cd* is raised. If rays higher than *a* or lower than *b* enter the block they become totally reflected, and pass out of the rhomb to one side or the other. The edge B, which is cut very fine and sharp, therefore serves as an excellent dividing line. The polarizing prism is placed in the beam *cd*, and the cell for holding any substance for test is placed in the beam *ab* above, in addition to a piece of neutral tinted glass intended to compensate for the loss of light in the Glan-Thompson prism.

In this spectrophotometer, as in other instruments of the same class, the possibility of polarization effects arising at glass-air surfaces is not to be forgotten. Hüfner in the original paper says: "Now in consequence of the magnitude of the angles of incidence under which the rays meet the rhomb, the rays *a* and *b* do not emerge without partial polarization. However, . . . this error is also eliminated

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[*] *Zeit. f. phys. Chem.*, Vol. III, p. 562, 1889.

(that is the fraction of polarized light mixed with the beam is again cut out) when it is ensured that the ray shall fall on the dispersion prism under the same angle of incidence as on the rhomb, the dispersion prism being made of the same glass and having its refracting angle turned at 90° to that of the rhomb. The angle of incidence is that which corresponds to the minimum deviation in the dispersion prism. If the angle of incidence i is known the required angles (B and D) for the rhomb follow; they are $2(90^\circ - i)$.

" In the apparatus here described the refracting angle of the dispersion prism is 60°, therefore the angle of refraction inside the prism is 30°, and since the refractive index for sodium light is 1·623, the corresponding angle of incidence is 54·24°. It follows that the rhomb must be so cut that the angle $B = 2(90^\circ - 54{\cdot}24^\circ) = 71{\cdot}52^\circ$ or $71^\circ\ 31'$."

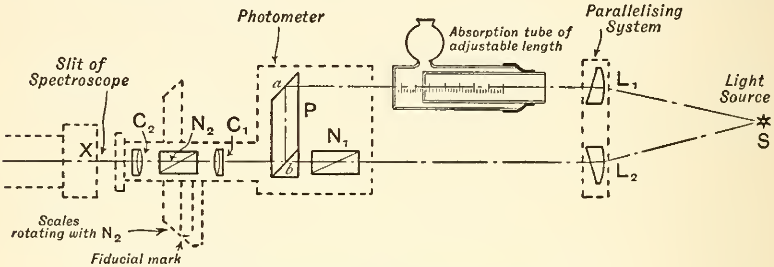

Fig. 151.—Optical System of Nutting Spectrophotometer

By some workers the Hüfner instrument is preferred to any other. It is fairly economical of light and simple in construction. The main disadvantages are that it only provides a simple bipartite field, which is undesirable (on account of eye fatigue in rapid work), and that it cannot conveniently be made up as an accessory to an ordinary spectroscope. Houston[*] has described a modification of the construction of the rhomb which is of some interest.

The Nutting Spectrophotometer was originally devised by Nutting, and is now made up in a convenient form by Messrs. Adam Hilger, Ltd. The instrument is shown in figs. 151 and 152. Light from the source S, usually a pointolite lamp of fixed focus type, is approximately collimated by the separated halves L_1 and L_2 of the split lens. These lens-prisms are mounted on a separate stand. The two beams then pass into the main part of the instrument, one of them traversing the polarizing prism N_1; they then reach the photometric prism P. In this prism the surface *b* is silvered in strips parallel to the plane of the diagram, so that, as viewed from the lens C_1, the surface is divided into horizontal strips illuminated

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[*] *Phil. Mag.* (6), 15, p. 282, 1908.

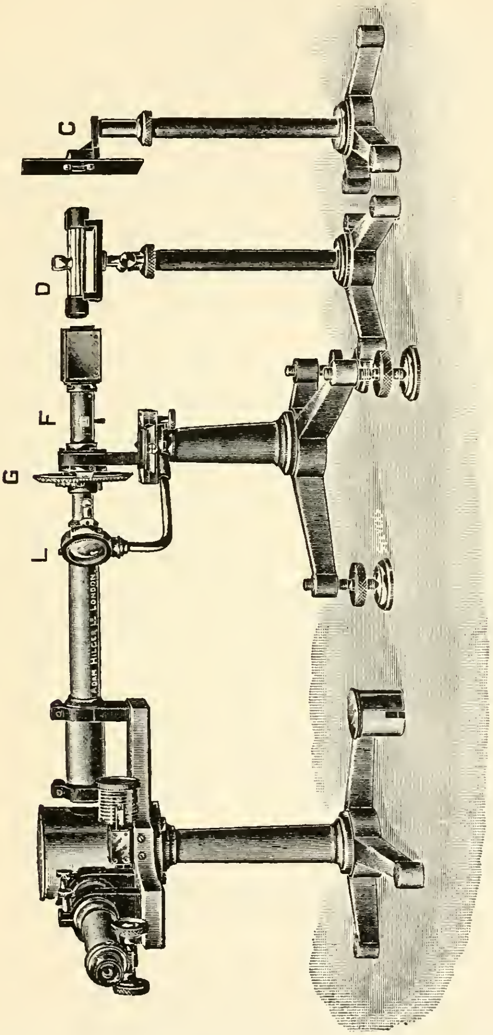

Fig. 152.—Nutting Spectrophotometer and Hilger Wave-length Spectrometer

L, Reading lens. G, Divided circle and density scale. F, Spectrophotometer tube. D, Absorption coil. C, Illuminating device.

alternatively by the two sources; the action is not unlike that of the photometer prisms described on p. 214. In the latest forms of the instrument the prism consists of a simple reflecting rhomb, which is so thin as to cover only the central part of N_1.

The light then passes on to the lens C_1, and traverses the analyser N_2 and the lens C_2. The two lenses form, in the plane X, an image of the separating surface *b* in the prism P. The image-forming rays are thus parallel in passing through N_2. The lens C_2 can, by means of a spiral focusing motion, be displaced parallel to its axis in order to effect the fine adjustment of the focus.

It is arranged that the image of the face *b* shall be formed on the slit of a spectroscope. Thus the spectrum seen in the field is divided into strips corresponding alternatively to the two beams. All the spectrum may be screened off except a narrow vertical strip for which a measurement is to be taken. Rotation of the analyser N_2 will only vary the intensity of the beam which is plane polarized when it arrives at the separating surface. It is therefore convenient to place a light filter for test in the upper beam in the diagram.

The use of two lenses C_1 and C_2, instead of a single lens as in the description of the original instrument, obviates any shift of the image (due to the turning of the analyser) which would be caused if N_2 were not quite correctly mounted. The rotation of the analyser is indicated by use of a divided circle and pointer.

Since the intensity of the variable beam is proportional to the square of the cosine of the angle through which the analyser is turned from the position of maximum brightness, it is convenient to place on the circle, as well as an angular scale, a scale of " densities " from which the density of a light filter may be read off directly. The definition of " density " is

$$\text{density} = \log_{10} \frac{\text{incident light}}{\text{transmitted light}}.$$

Therefore the density number corresponding to an angle θ will be

$$D_\theta = \log \frac{I_0}{I_0 \cos^2\theta} = \log \sec^2\theta.$$

When $\theta = 0$ the density will be the logarithm of 1, which is 0. When θ is 90° the density will be " infinity ". It is usually arranged that the parts of the field are equally bright when θ has a small finite value, θ_0 say, the variable part having been a little reduced. When the filter has been placed in the non-varying beam the angle for matching is θ_1, say.

Hence the proportion of light transmitted by the filter is given by

$$\frac{\text{incident light}}{\text{transmitted light}} = \frac{\cos^2\theta_0}{\cos^2\theta_1}.$$

Therefore the density of the filter is log $\sec^2\theta_1$ — log $\sec^2\theta_0$, which is the difference of the two readings on the density scale.

The spectrum is in focus in the field of view, but as measurements will be made at a limited number of points in the spectral range it is usually arranged that the width of the field, and therefore the range of wavelength viewed at any point, shall be limited by a diaphragm of adjustable width placed in the focal plane of the eyepiece. This is a great convenience when light filters having narrow absorption bands are to be examined.

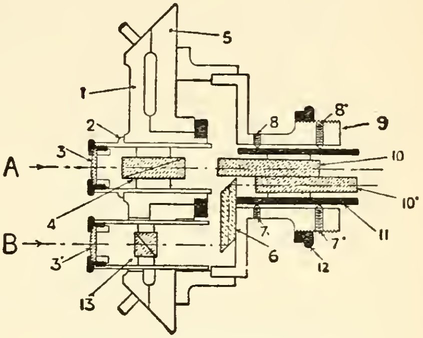

Fig. 153.—The Bellingham-Stanley Spectrophotometer

The Bellingham-Stanley Spectrophotometer.—Messrs. Bellingham & Stanley, Ltd., have recently introduced a spectrophotometer which is illustrated in fig. 153. It is intended for clamping against the slit of a spectrometer, the tube (11) being brought flush with the jaws of the slit. A polarizing prism (4) in the tube A is mounted in a rotating part (1) of which the rotation (giving either " degrees " or " densities ") can be measured. The tube B does not rotate; it contains a double image compensating prism. The source of light is placed on the axis A, and the lens (3) is of such a power that it forms an image of the source on the telescope objective of the spectrometer; the light passing to the tube B is also brought to the slit by the aid of the specially formed glass prism (6), the action of the lens (3′) being similar to that of (3). The analyser is formed of two calcite plates (10 and 10′) cut in a manner to that described on p. 242, except that the ends are trimmed square. Each plate allows only the extraordinary ray to escape, hence the intensity of the upper beam, but not the lower, will vary with the rotation of (4). Evidently at maximum brightness the variable beam should match the constant one. The upper sharp edge of (10′) forms the separating line in the field of view.

Since there are no glass surfaces between polarizer and analyser in the variable beam, the possibility of errors due to elliptically polarized light is precluded. The least satisfactory feature about the instrument will probably be the difficulty of keeping the edge of (10′) in good condition if the instrument has frequently to be removed from the spectrometer. Apart from this the arrangement has (*a priori*) several points in its favour.

The use of double image prisms in which the parts are placed in optical contact, or in which Canada balsam is replaced by glycerine, enables the system to be used for ultra-violet photometry by photographic methods.

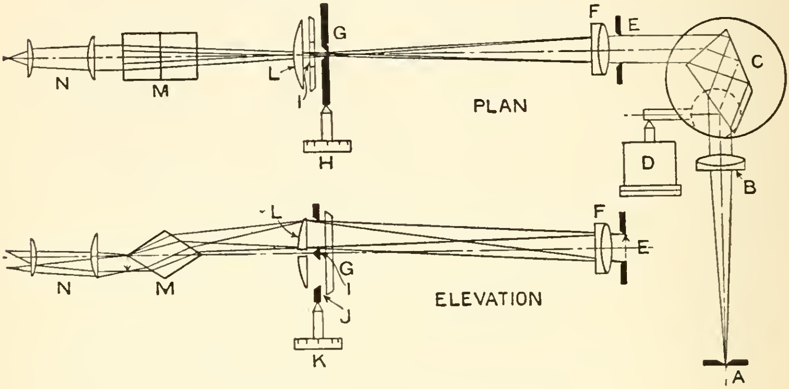

Fig. 154.—Bull's Spectrophotometer

Bull's Non-polarizing Spectrophotometer.*—Several of the considerations involved in the use of variable apertures in photometry are well illustrated by Bull's spectrophotometer, which is chosen for description here because of that consideration. It is important that entire reliance shall not be placed on the indications of any one type of instrument when results are required which are to have any pretensions to absolute accuracy. The spectrophotometer is illustrated in plan and elevation in fig. 154. By the aid of a prism placed over one half of the slit of the Hilger "wave-length spectrometer" shown in the right-hand parts of the figure (with its slit A, collimator and telescope lenses at B and F respectively, and the constant deviation prism C), the two parts of the slit can be

* *Trans. Opt. Soc.*, Vol. XXIII, No. 3, p. 197.

illuminated by different sources. The lenses B and F produce an image of the slit in the plane of the aperture G. The width of this aperture can be controlled by the micrometer head H, and the range of wave-length which it transmits from the spectrum is evidently dependent on this width. Referring to the elevation, it will be seen that the *upper* and *lower* parts of the same aperture are illuminated respectively by light from the *lower* and *upper* parts of the slit. The lens L, which is in two parts slightly separated, is of such a power that it forms two images of the aperture E, and these are brought into juxtaposition by the rhomb M, the action of which is suggested by the ray paths drawn in the diagram. Since the lens L has only a narrow vertical strip exposed, these images of E are in monochromatic light. The intensity of illumination of the upper part of the field, as seen in the eyepiece M, can be varied by the action of the micrometer head K which controls the *length* of the lower part of G; the intensity is taken to be proportional to this length, a supposition which is doubtless accurate except when the length is very small. In the latter case diffraction will result in the spreading of some of the light out of the proper image, and the intensity law will no longer hold. Special precautions are necessary in securing the uniform distribution of light into the collimator of the instrument, and Bull describes the use of the property of uniform reflection from the interior surface of a matt white hollow sphere as a successful means of securing this end. Instead of illuminating the two parts of the collimator slit by two different sources of light, the instrument may be arranged to examine the selective transmission or reflection of some substance by illuminating the two parts of the field by the same source, and arranging the test substance so that light has to be transmitted through it or reflected from its surface before entering the lower portion of the slit. In cases where reflected light has to be examined it is usually difficult to secure an intensity of illumination sufficiently intense for comfortable work with spectrophotometers. The measurement is usually made with greater convenience with the aid of the colour-patch apparatus. For a description of this and of a special illuminating fitting intended for use when opaque materials are to be examined with a spectrophotometer, the writer's *Colour and Methods of Colour Reproduction* may be consulted.

Stray Light in Spectrophotometry.—Stray light in connection with photometry has already been mentioned. Its effect on spectrophotometric measurements is often very considerable, as is suggested by the provision of special prisms for its elimination in the König-Martens spectrophotometer. In some cases the stray light takes a path perceptibly different from that of the main beam, and consequently the hues of direct and stray light in the field of view are not identical. In such a case a colour filter which transmits a narrow range of wave-length, in the neighbourhood of the wave-length of the light being employed, may do much to minimize the relative brightness of the stray light. Alternatively in some cases, a direct-vision prism may be mounted behind the eyepiece; this should produce its dispersion in a direction perpendicular to the length of the original spectrum, and it will thus deviate much of the stray light. Devices of this kind, however, have to be employed with caution, as anything which tends to reduce the brightness of the field, or to make it more awkward to view, is very likely to reduce the accuracy obtained.

Impurity of Spectrum.—In some types of spectrophotometer such as the Nutting, the actual spectrum is in focus in the field of view, and any spectral lines or absorption bands are clearly seen. It is somewhat difficult to measure the true variation in brightness through an absorption band, because if the field is made so narrow as to include only a small part of the band it is often too narrow for satisfactory matching. On the other hand we have to consider spectrophotometers like the König-Martens or the Bull, in which the field is uniform, but the light reaching the eye corresponds to a definite finite spectral range on account of the finite width of the aperture through which the observation is made. These considerations apply also to photoelectric and other non-visual photometers. Martens and Grünbaum estimate for the König-Martens instrument that if the objective or collimator slits are 0·1 mm. wide and the ocular slit is 0·25 mm. the field only covers about 4 $\mu\mu$. Such fine slits would, however, be rather difficult to use in all circumstances. The effect of spectral impurity was discussed by Rayleigh,* and we may usefully introduce a discussion on similar lines.

Imagine a " pure " spectrum to be focused on a screen and let the true intensity be regarded as some function of the linear displacement; it is represented by some curve PRQ (fig. 155) in which the ordinates represent the intensity y (defined as the differential coefficient $\frac{dE}{dx}$, where E represents energy).

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* *Phil. Mag.*, Vol. XLII, 1871, p. 441.

Now imagine that the slit of the spectroscope is widened so that its monochromatic image is of width $2k$ at the point x. The radiations, most widely divergent in wave-length, which fall on an elementary strip of infinitesimal width at x will be those corresponding in the pure spectrum to $x - k$ and $x + k$.

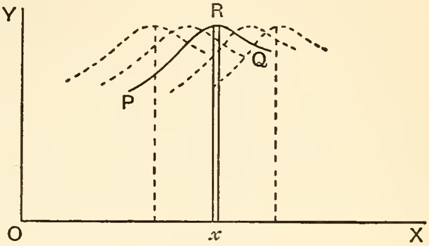

Fig. 155

The energy falling on a very narrow elementary strip of unit width will thus be proportional to

$$y_1 \text{ (say)} = \int_{x-k}^{x+k} y dk.$$

We may *define* y_1 as the intensity of the impure spectrum; hence the meaning of the term intensity is fixed. Note that if the broadening of the *image* had been due merely to optical diffuseness without a corresponding increase in the width of the slit itself the intensity would have been given by

$$y_2 = \frac{1}{2k} \int_{x-k}^{x+k} y dk.$$

This is the case discussed by Lord Rayleigh.

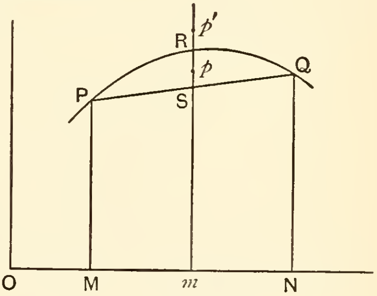

Fig. 156

Let it now be supposed that there exists in the screen an aperture of width $2h$, which has its length parallel to the monochromatic image of the slit. Let the curve PRQ (fig. 156) represent in a similar way the intensity of the *impure* spectrum with the widened slit. M and N are the two points with an interval $2h$, and m is the mid-point.

(In the instrumental application this aperture MN might represent the ocular slit for observation in such instruments as the König-Martens spectrophotometer, or the aperture placed before the bolometer or thermopile in an infra-red apparatus.)

Now p is a point on the mid-ordinate such that

$$pm \times \text{MN} = \text{area MPRQN};$$

let y' be written for pm, then y' corresponds to the mean intensity which would be found by integrating the radiations passing through the aperture of width $2h$.

The area

$$\text{MPRQN} = \int_{x-h}^{x+h} y_1 dh,$$

hence

$$y' = \frac{1}{2h}\int_{x-h}^{x+h}\int_{x-k}^{x+k} ydkdh. \dots\dots\dots\dots\dots (1)$$

This may be evaluated using the method employed by Lord Rayleigh. Consider the integral

$$\int_{-h}^{+h} ydh.$$

Putting $y = f(x_0 + h)$ we have, by using Taylor's theorem and neglecting terms of higher order than those involving h^2,

$$\int_{-h}^{+h} ydh = \int_{-h}^{+h}\left(y_0 + \frac{dy}{dx_0}h + \tfrac{1}{2}\frac{d^2y}{dx_0{}^2}h^2\right)dh$$

$$= 2h\left(y_0 + \frac{h^2}{6}\cdot\frac{d^2y}{dx_0{}^2}\right)$$

(the differential coefficients given subsequently refer always to the abscissa given by $x = x_0$); thus

$$\frac{1}{2h}\int_{x_0-h}^{x_0+h} ydh = y_0 + \frac{h^2}{6}\cdot\frac{d^2y}{dx^2} \dots\dots\dots\dots\dots\dots (2)$$

and

$$\int_{x_0-k}^{x_0+k} ydk = 2k\left(y_0 + \frac{k^2}{6}\cdot\frac{d^2y}{dx^2}\right) \dots\dots\dots\dots (2a)$$

Differentiating the last expression twice and multiplying through by $\frac{h^2}{6}$ we obtain,

$$\int_{x_0-k}^{x_0+k}\frac{h^2}{6}\cdot\frac{d^2y}{dx^2}\cdot dk = 2k\frac{h^2}{6}\left(\frac{d^2y}{dx^2} + \frac{k^2}{6}\frac{d^4y}{dx^4}\right) = \frac{2kh^2}{6}\cdot\frac{d^2y}{dx^2},$$

if the term in h^2k^2 may be neglected. Thus,

$$y' = 2k\left(y_0 + \frac{h^2 + k^2}{6}\cdot\frac{d^2y}{dx^2}\right) \dots\dots\dots\dots\dots (3)$$

In the case where the image is diffuse without the brightening due to the widening of the slit we shall have

$$y' = y_0 + \frac{h^2 + k^2}{6} \cdot \frac{d^2y}{dx^2},$$

$$\text{or} \qquad y_0 = y' - \frac{h^2 + k^2}{6} \cdot \frac{d^2y}{dx^2} \text{ (Rayleigh's case)} \ldots\ldots (3a)$$

To pass from the observed curve to the true we may use the following argument.

Suppose that y can be represented by the expression

$$y = \mathrm{F}(x_0 + \sqrt{h^2 + k^2}).$$

Referring to fig. 156, imagine that the curve PRQ is the true (pure spectrum) intensity curve and that M and N are at distances $\sqrt{h^2 + k^2}$ from m, the supposed position of the centre of the slit. Then

$$\mathrm{S}m = \tfrac{1}{2}\left[\left\{y_0 - \frac{dy}{dx_0}\sqrt{h^2 + k^2} + \frac{d^2y}{dx^2}\left(\frac{h^2 + k^2}{2}\right)\right\} + \left\{y_0 + \frac{dy}{dx_0}\sqrt{h^2 + k^2} + \frac{d^2y}{dx_0^2}\left(\frac{h^2 + k^2}{2}\right)\right\}\right]$$

if higher order terms be neglected.

From the above we find

$$\mathrm{S}m = y_0 + \frac{d^2y}{dx^2} \cdot \frac{h^2 + k^2}{2}$$

$$\text{and} \qquad \mathrm{RS} = y_0 - \mathrm{S}m = -\frac{d^2y}{dx^2} \cdot \frac{h^2 + k^2}{2}.$$

In the latter case (Rayleigh's) we shall then have

$$y_0 = y' + \tfrac{1}{3}\mathrm{RS}, \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

but in the former and more usual case we shall have

$$y_0 = \frac{y'}{2k} + \tfrac{1}{3}\mathrm{RS}$$

(ordinary impure spectrum with collimator and ocular slits of finite width), or

$$2ky_0 = y' + \frac{2k}{3} \cdot \mathrm{RS} \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Now in Rayleigh's case we find, by differentiating equation (3*a*) above,

$$\frac{h^2+k^2}{6}\cdot\frac{d^2y}{dx^2} = \frac{h^2+k^2}{6}\left(\frac{d^2y'}{dx^2} - \frac{h^2+k^2}{6}\cdot\frac{d^4y}{dx^4}\right) = \frac{h^2+k^2}{6}\cdot\frac{d^2y'}{dx^2},$$

if terms involving h^4 and others of the same order be neglected. In order to be able to pass from the observed curve to the true it is necessary that h and k shall be small enough to render this supposition valid. The curvatures of the observed and true curves become very nearly equal, and the RS of the observed curve is equal to the RS of the true one in Rayleigh's case. It must not be considered that the true curve passes through points P and Q on the observed curve. It is necessary to plot a number of points by the following rule in order to obtain the true shape of the correct curve. This point is not made quite clear in the original paper.

The principal rule which Rayleigh gave is as follows. Construct the curve representing the immediate results of the observation in the usual way. Let Rm be any ordinate. Draw parallels PM, QN at distances from m equal to $\sqrt{h^2+k^2}$ and join PQ cutting Rm in S. The point p', on the true curve, corresponding to the abscissa Om is to be found by taking p'R equal to $\frac{1}{3}$RS, so that p' and S lie on opposite sides of R.

That the *rule* is the same in the second case (equation 5) will appear from the fact that the ordinates of the observed curve are in general about $2k$ times those of the true curve, and hence the RS of the observed curve requires a correction $2k$ times the $\frac{1}{3}$RS of the true curve. Hence the form of the rule need not be altered.

The original curve will have ordinates which, in "flat" parts of the curve, are very nearly equal to the true values, and the performance of the instrument in the neighbourhood of sharp absorption bands, &c., can be investigated by the method just described. It will be understood that this method is immediately available for correcting transmission curves obtained by spectrophotometers, and for assessment of the amount of error in cases where the rule is not exactly applicable.

Necessity for Checking Results.—If great accuracy is desired, two or more differing methods or instruments should be employed for the same measurement. Sometimes a number of points on a curve can be checked by employing monochromatic light with a simple photometer. Such sources are a sodium flame or a mercury lamp used with suitable light filters (supplied by Messrs. Kodak, Ltd., Wratten Division) for isolating the brightest lines of the mercury spectrum.

Stereoscopy and its Applications to Photometry.[*]—During recent years a number of original methods of photometry have been developed in Germany, springing from the chance observation of an unusual effect in the use of the stereo-autograph.[†] The nature of the effect may be demonstrated as follows. Suspend a pendulum bob by two threads, say 4 ft. in length, from a horizontal bar, the ends of the threads being tied about 2 ft. apart. The bob then swings accurately in one plane, the trace of which (PP_1) is shown diagrammatically in fig. 157. Just below the track of the bob is placed a reference pointer *r*. Both bob and pointer may be silhouetted against a fairly bright background. The pendulum is given an amplitude sufficient to make it move through the lowest point of swing with fair velocity, and it is then observed by both

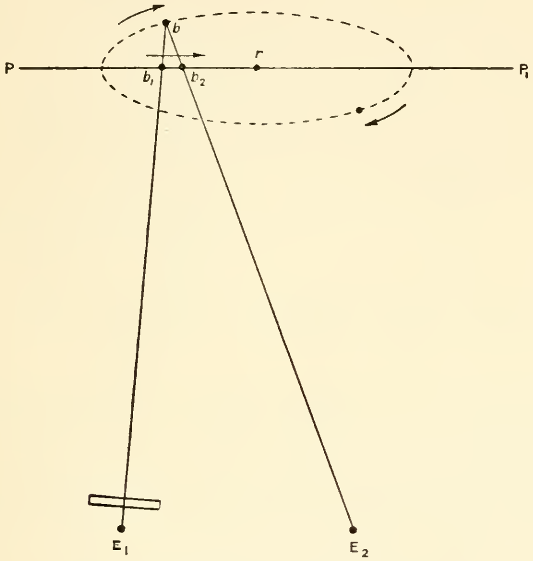

Fig. 157.—Principle of the Displacement Effect

eyes; the left eye, however, is covered by a piece of "neutral tint" glass such as is used for spectacles. (For a simple demonstration almost any piece of "smoked" or coloured glass will give the effect.) To a person with good stereoscopic vision the pendulum then appears to swing in a more or less elliptical path. This is explained by a relative time lag in the sense impression reaching the brain, caused by the diminished intensity of the light stimulus acting on the retina of the left eye. When the bob moves to the right, one sees the bob, by using the right eye, apparently in the direction of b_2 at the same time that it is seen by using the left eye in a direction farther behind. The "mental convergence" to which reference was made in the brief discussion of stereoscopy previously given (Chapter VII) may now be imagined to come into play, so that the objective is located in the sense-space-picture at the point *b*. When the light filter is placed before the right eye the sense of the revolution is reversed.

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[*] C. Pulfrich, *Die Stereoskopie im Dienste der Photometrie und Pyrometrie* (J. Springer, Berlin, 1923).

[†] See *Photography as a Scientific Implement* (Blackie), p. 393, also books by C. Pulfrich, *loc. cit.*, p. 115.

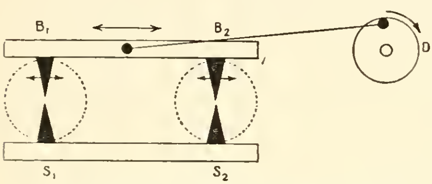

Fig. 158.—The Fixed and Moving Marks seen in the Field View of each Telescope

The nature and amount of the time lag were investigated (many years ago) by S. Exner,* with the assistance of v. Helmholtz, and this research showed clearly that the growth of the sense impression of light takes place more quickly the stronger the stimulus. The explanation of the stereoscopic effect is thus so far complete. The application to photometry can now be understood.

The basis of the stereoscopic photometer is generally a binocular telescope, not of the Galilean variety, but of the type possessing positive eyepieces. The arrangement shown diagrammatically in fig. 158 is placed in the common focal plane. S_1, S_2 are two stationary pointers immediately underneath, and in the same plane as, two moving pointers B_1, B_2 which are fastened to a sliding carriage made to move backwards and forwards like a piston by the revolution of the disc D. The distance between B_1 and B_2 is exactly equal to that between S_1 and S_2. Consequently the stereoscopically fused images of the two pairs of pointers appear at the same distance when B_1 and B_2 are still. Fig. 159 represents a simple case in which it is desired to compare two light sources l_1, l_2. These, by the aid of mirrors m_1, m_2, illuminate the equal discs of opal glass placed in front of the " object glasses " of the telescopes. In each section, therefore, the field is of comparatively uniform brightness, and when both fields are equally bright the moving mark seen on looking into the instrument with both eyes will appear to remain in the same plane as the stationary mark. When, however, the fields are not equally bright, the appearance of the elliptical motion of the moving pointer is set up. In the present case the fields could be equalized in brightness by displacement of the mount carrying the two lamps, and the intensity ratio could be calculated from the " inverse square " law. Certain personal and instrumental errors will be removed if another result (obtained by reversal of the lamps so that l_1 is nearer m_2 and vice versa) is given equal weight in the determination.

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* *Sitz. Wein. Akad.*, Vol. LVIII, 1868.

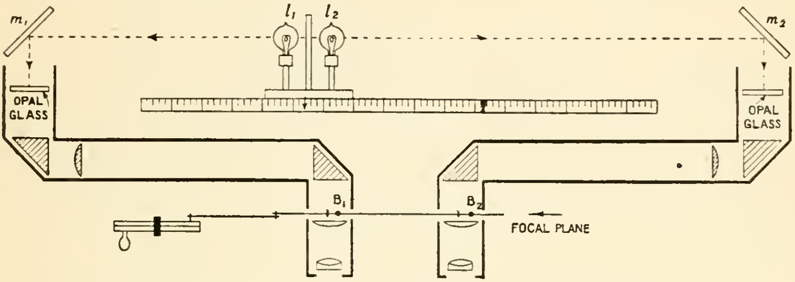

Fig. 159.—Stereoscopic Photometer used to compare two Lamps

This is one of the simplest arrangements which Pulfrich describes. The same principle is applied to photometers of many kinds, including spectrophotometers. The principle affords another method of overcoming some of the difficulties of heterochromatic photometry. At the same time, however, personal and psychological factors have a great influence on the results, and it is not quite clear whether the real " brightness function " can thus be measured. *Experientia docet.*

Photometric Settings.—The method usually adopted in photometric settings is to vary the contrast in steps of diminishing amplitude on each side of the true match. The final setting of the moving part being judged as midway between its positions for just perceptible contrast on either side. The eye should be rested every few seconds. The moving parts of the apparatus must work smoothly and easily, and the greatest attention must be given to the physical comfort of the observer. Noisy rotating parts or other " worrying " things should be avoided whenever possible.

CHAPTER XIII

Polarimeters and Saccharimeters

Rotation of the Plane of Polarization.—One of the most important of the optical instruments used in industry is the *polarimeter*; the *saccharimeter* is a specialized form. In common with certain other so-called " optically active " substances, cane-sugar in solution has the property of rotating the plane of polarization of plane polarized light during transmission, and this action is employed for the estimation of the amount of sugar in a solution.

The conventional terminology, due to Biot, calls a clockwise rotation, *seen when facing the source of light*, a positive or right-handed rotation and vice versa. Substances vary in the direction and magnitude of the rotation they produce; the varieties of sugar *dextrose* and *lævulose* produce, as the names suggest, right-handed and left-handed rotations respectively.

Within certain limits, and in comparatively dilute solutions, the rotation produced is very nearly proportional to the concentration of the solution, but it varies with the temperature. The different wave-lengths of white light also undergo varying amounts of rotation; thus we have " rotary dispersion ". As in ordinary dispersion, the greater effect is produced on the shorter wave-lengths.

The rotation produced by a decimetre length of a solution, divided by the weight of the active substance in unit volume of the solution, is known as the *specific rotation.*

Schönrock* gives the following formula for the specific rotation of cane-sugar for sodium (D) light at a temperature t° C.:

$$[\alpha]_{D}^{t} = 66{\cdot}5^\circ - 0{\cdot}0184\,(t - 20).$$

In this case the specific rotation is nearly independent of the concentration, but a greater variation due to that cause is found in the case of " invert " sugar or " lævulose ".

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* *Zeit. Instrum.*, 1900, Vol. XX, 97.

Rotation produced by Quartz.—It will be remembered that quartz plates cut perpendicular to the axis of the crystal have the property of rotating the plane of polarization in a similar manner. Some crystals produce right-handed and others produce left-handed rotation, thus giving rise to the names of the two varieties of quartz.

Determination of the Plane of Polarization.—The simplest means of determining the plane of polarization of light (in order to detect and measure any rotation of that plane) is obviously the use of the simple Nicol prism, which can be turned until the light is extinguished. We may thus arrange a sodium flame, a polarizing prism, a tube of solution, and an analyser suitably mounted with a divided circle as the very simplest type of polariscope. There might be, however, a short range of angular positions for the analyser in which the transmitted light does not reach the "threshold" brightness of vision (L_0) of the human eye, and would not be visible.

The maximum error of setting will therefore be given by some angle U, where we have, if U is the angle turned from the physical extinction point,

$$L \sin^2 U < L_0,$$

where L is half the intensity of the source, the maximum intensity of the field. Now according to Nutting's results, L_0 is of the order of 16×10^{-6} millilamberts, and therefore if we require to bring down U to 0·01°, say, the brightness L would have to be increased to 200 millilamberts, which would be difficult with the ordinary means of producing sodium light, and moreover the conditions encountered in practice would not permit of that very low value of the threshold being valid. The original instruments developed by Biot and Ventzke, and introduced about A.D. 1840 had this simple device, but they were soon found too insensitive; consequently Soleil, Robiquet, and others devised instruments in which an equality setting, similar to that of a photometer, could be substituted for the extinction setting.

One of the earliest devices of this kind was to use in place of the simple analyser a "Jellet-Cornu prism", to make which a Nicol prism is cut longitudinally; wedge-shaped pieces are ground off each half and the two halves are cemented together once more, as indicated in fig. 160.

The arrangement gives an angle of about 5°, according to the construction, between the planes of polarization of the light from the two halves of the prism. Such a prism was used in the " Duboscq *half-shade* rotatometer ".

Another device described somewhat later by Lippich* places a second small Nicol prism behind the polarizing prism in the instrument; this has its principal section inclined at a small angle to that of the main prism. Two of such small auxiliary prisms of the square ended type, kept parallel or nearly parallel to each other, are used in modern instruments, and the field is thus divided into *three* parts (fig. 161).

In either case it will be seen that light from both sections of the field will not be simultaneously extinguished. There will, however, be one point at which the intensity of light from both parts is equal, and this condition provides a means of making a better setting of the analyser. When using monochromatic light, the rotation produced by placing a tube of an optically active substance in the path of the light affects light from both halves equally; any means by which the analyser can be consistently placed in a definite angular position with regard to the plane of polarization of the original light can be used for a measurement of rotation.

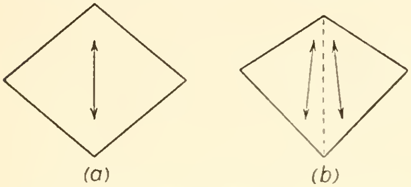

Fig. 160.—End views of (*a*) Nicol Prism, (*b*) Jellet-Cornu Prism. The arrows indicate the direction of vibration of the polarized light.

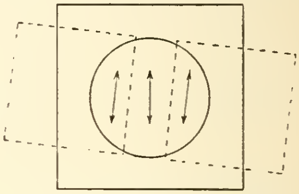

Fig. 161.—Diagram to illustrate production of Tripartite Field in a Polarimeter by use of two Auxiliary Prisms (Lippich end-point device).

As was remarked in the chapter on photometers, the oblique transmission of light through reflecting surfaces is liable to cause slight effects of elliptical polarization. Consequently it is usual in modern instruments to employ polarizing prisms such as the Glan-Thompson, in which the end faces are normal to the transmitted light. This applies also to the auxiliary prisms employed in the Lippich end-point device. Nevertheless in the auxiliary prisms one or two special requirements must be fulfilled. A plan of the device is given in fig. 168. The edges *e* of the auxiliary prisms must be perfectly sharp and free from flaws, and in order to get a perfect separating line in the "field" the angle of the prism at this edge has to be 92° to 94°, i.e. slightly more than the right angle. The auxiliary prisms can then be slightly tilted so that rays, imagined incident just inside the critical edge of the auxiliary prism, from any point of the analyser diaphragm, can be transmitted through without obstruction.

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* *Zeit. f. Inst.*, **2**, 167; **14**, 326.

Lippich showed that when two auxiliary prisms are employed and placed so that their principal sections are very slightly inclined one to the other, the arrangement is more sensitive for detecting differences of brightness between the centre and one or other of the outer parts of the field. With the most sensitive conditions and a bright source of light, the mean error of readings may be reduced as low as ± 15″ of arc, but this is only obtained when the light is bright enough to enable the half-shadow angle to be reduced to 1°.

Another arrangement which is employed as an end-point half-shade device was introduced by Laurent in 1877, and employs a plate of quartz cut parallel to the optic axis. It must be of such a thickness that one of the components into which a plane polarized wave, passing normally through the plate, is resolved, gains an odd number of half wave-lengths over the other. The necessary thickness can be found from the expression

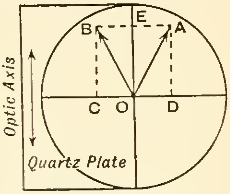

Fig. 162.—Laurent's Half-shade

$$d(N_e - N_o) = (2m + 1)\frac{\lambda}{2},$$

where d = thickness of plate, N_e and N_o are extraordinary and ordinary refractive indices, and m is an integer; d is made at least 0·5 mm. in practice.

In the diagram (fig. 162) the quartz plate covers half the field of view, its edge OE being parallel to the optic axis. The original plane vibration from the analyser represented vectorially by OA can be resolved, in the left-hand half of the field only, into two components represented by OE and CD, parallel and perpendicular respectively to the optic axis. During the passage through the plate the vibration parallel to OD gains half a wave-length on that parallel to OE. We therefore represent the emergent components by OC and OE, which combine to form the plane vibration OB. It is not necessary that the optic axis of the plate shall be strictly parallel to OE, but the directions of vibration in the parts of the field will always be symmetrical with regard to the axis.

We thus have, as in the two former cases, distinct parts of the field giving light polarized at slightly different angles. In this case the angle obviously depends on the original inclination between OA and OE (or the direction of the axis), which can be adjusted to any required value.

Several criticisms, however, can be brought forward with regard to the Laurent instrument. The action of the half-wave plate depends entirely on the absolute homogeneity of the crystal and the plane parallelism of the faces, both of which are difficult to get entirely perfect. Also the light passing through the plate should be strictly parallel, but since it is generally convergent to some slight extent there may arise small amounts of elliptical polarization, and the transmitted light will not then be uniformly plane polarized.

It was proved by Lippich that several Laurent instruments would differ in their own indications on one test substance, and also from a Lippich instrument by amounts varying from a few seconds to over a minute. Hence it may be concluded that the Laurent half shade is unsuitable for work of the highest accuracy. The mean error of adjustment, even under good conditions, may be as high as $\pm 2'$ of arc.

Bellingham and Stanley's Half-shade Device.—Comparatively recently (1922) Bellingham and Stanley have introduced a saccharimeter in which a novel end-point device is employed. It had been pointed out that in the usual form of the Lippich device the auxiliary prisms are constructed so that the prism faces are not the natural cleavage planes of the crystal (Iceland spar), and the sharp edges are therefore not in the natural crystal faces. This was given as the reason for the disintegration of the edge which has been sometimes noticed, for it is well known that natural edges and crystal faces are the most stable in a mechanical sense. Bellingham and Stanley employ a polarizing prism constructed from two solid rhombs of the natural spar. " Each is so cut with respect to the crystallographic axis, and is of such a length of side, that only the extraordinary ray passes out through the prism, the ordinary ray being absorbed at the side. . . . The face of each of the rhombs is a natural face of the crystal, and one of them contains the sharp sensitive edge."

Referring back to the description of the Jellet-Cornu prism, the manner in which the two natural rhombs are then treated in order to obtain the half-shade angle will be understood; the side of each is ground away to the required extent, and the two parts are then held together in contact. Fig. 163 illustrates the arrangement. In fig. 164 the arrows show the vibration direction of the extraordinary ray, i.e. in the plane which contains the crystal axis and the shorter diagonal of the end faces. The two rhombs are seen to be displaced relatively to one another; this is to secure that one edge only shall be in focus when viewed by the observing telescope, and shall thus form the line of division in the field of view. When the setting has been made the edge is practically invisible. It will be noticed that this does not give the tripartite division or contrast pattern considered by some to be necessary for the greatest sensitiveness in setting, but the arrangement has several compensating advantages, amongst which are:

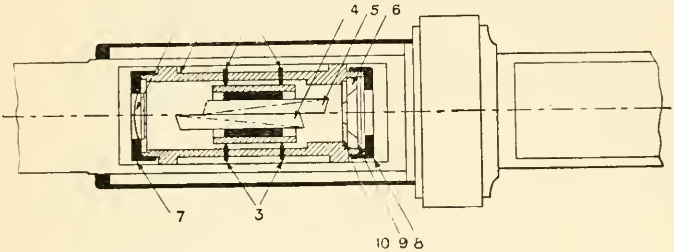

Fig. 163.—Bellingham and Stanley's Half-shade

1, Condensing lens. 3, Adjustment for prism system. 4, 5, Prism components. 9, Protecting window. 10, Diaphragm.

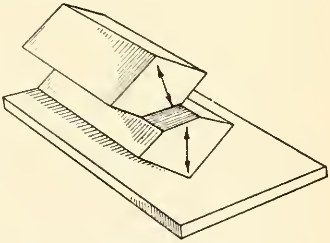

Fig. 164.—Method of Construction of Bellingham and Stanley's Half-shade Prism

1. *Simplicity.* Two rhombs of spar (uncemented) are employed, as against three polarizing prisms required in the Lippich device.

2. *Stability.* (This has already been discussed.)

3. *Possibility of use for investigations with ultra-violet light.* The film of Canada balsam in an ordinary cemented prism is quite opaque to the shorter ultra-violet waves, but by employing the above prism, and quartz lenses and plates of right- and left-handed quartz components (each cut to equal thickness so that their rotation is eliminated), investigations of the rotary powers of various substances for ultra-violet light can conveniently be carried out. In the mounting

adopted by Bellingham and Stanley the prism is protected on one side by a lens and on the other side by a glass plate.

Tests for the Mounting of the Prisms.—Two very general tests, which may be applied to most polarimeters, for the correctness of the mountings of the prisms are as follows:

1. The two zero points of the apparatus must be *exactly* 180° apart.

2. If a large angle, say nearly 90°, is measured, the two final readings, 180° apart, must give exactly the same value for the angle of rotation.

Sources of Light for Illumination.—It is extremely desirable to secure as bright a field of view as possible. The usual " sodium flame " in the Bunsen burner is hardly satisfactory in this respect as a source of light, but an improvement may be made with the " Meker " burner where the flame is hotter. Asbestos saturated with common salt is convenient to hold into the flame. Preparations containing sodium bromide give a brighter flame, as this salt is more volatile, but care must be taken to avoid the fumes to which it gives rise. When very great intensity is required a flame fed with oxygen must be used. A " borax bead " makes a convenient means of procuring the sodium light. Lippich suggests as a light filter a 10-cm. length of 6 per cent potassium bichromate solution in series with a cell containing uranous sulphate produced according to special directions (quoted in full by Landolt).* For most purposes, however, the bichromate solution will be sufficient, provided precautions are taken.

It has been shown that the " centre of gravity " of the double sodium line remains sufficiently nearly the same through the variations of intensity likely to be ordinarily used, but the method of filtering or purification does exercise, according to Landolt, a considerable influence on the optical centre of gravity of the light. Soret and Sarasin,† also Lippich,‡ have shown that for a quartz plate 1 mm. in thickness, the difference in rotation of the two D lines (0·6 $\mu\mu$ variation in wave-length) is 2′ 40″. According to Dietrich the ratio of the intensities of the two sodium lines is given by

$$\frac{D_2(589{\cdot}02\ \mu\mu)}{D_1(589{\cdot}62\ \mu\mu)} = 1{\cdot}6,$$

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* *Optical Rotation of Organic Substances*, p. 402 (The Chemical Publishing Co., Easton, PA., U.S.A.).

† *Compt. rend.*, **95** (635), 1882.

‡ *Wein. Sitz.*, **11**, 99, 722, 1890.

which gives the theoretical centre of gravity for the fully purified light as 589·25. Landolt has given the results of some observations on the optical centre of gravity of sodium light produced in various ways. He first measures the rotation for homogeneous light (D_2), then for the special light. Taking the wave-length of D_2 as above and assuming a change of 2′ 40″ of rotation per 0·6 $\mu\mu$ variation of wave-length, the effective wave-length of the special light is then calculated. The following extracts from his table may be given:

Source of Light.	Purification.	Wave-lengths in $\mu\mu$ of Effective Optical Centre of Gravity.
Landolt's sodium lamp with NaCl.	Not purified.	588·06
Bunsen burner with NaCl.	Layer of a 9 per cent aqueous solution of $K_2Cr_2O_7$ 10 cm thick.	589·48
Sodium light.	Perfectly purified (2 sodium lines only).	589·25
Bunsen burner with NaBr.	Layer of 9 per cent aqueous solution of $K_2Cr_2O_7$ 10 cm. thick.	592·04

In statements of rotations for sodium light it is therefore very important to give the centre of gravity. Otherwise, according to Landolt, the actual rotations may apparently be uncertain to 1 per cent or more.

In view of the uncertainty likely to be encountered in working with sodium light, most modern precision work in polarimetry is carried out with the green line (546·1 $\mu\mu$) of the spectrum of the mercury arc. Since there are apparently no light filters capable of isolating the radiation in sufficient purity, it is obtained pure by means of a monochromator (a spectroscope in which a slit replaces the eyepiece). The rotary powers of quartz for sodium and mercury light at 20° C. have been measured, giving the result:

$$\frac{\phi_{\lambda=589\cdot25}}{\phi_{\lambda=546\cdot1}} = 0\cdot85085.$$

Even in series with the monochromator, a green filter will be useful in removing stray light. The mercury arc is conveniently given by one of the several forms of small fused quartz lamps now obtainable, such as the "K.B.B." lamp (of Messrs. Kelvin, Bottomley, & Baird, Ltd.), with which a convenient mounting and hood can be obtained to protect the eyes from the dangerous ultra-violet portion of the radiations.

Sensitiveness of Equality Setting.—Using the symbols L and U in the same sense as that on p. 239, we have

$$B = L \sin^2 U,$$

where B is the intensity of the transmitted light in each half of the field;* hence

$$\frac{dB}{dU} = L \sin 2U.$$

When $U = 0$ or very small, $\frac{dB}{dU}$ becomes very small also. The difference of brightness between the two parts of the double field in the later types of instrument will be equal to

$$\delta B = 2L \sin 2U \delta U,$$

where δU is the angle of departure from the equality setting, and therefore

$$\frac{\delta B}{B} = \frac{2L \sin 2U \delta U}{L \sin^2 U} = 4 \cot U \delta U.$$

The value of $\frac{\delta B}{B}$ will be determined by the sensitiveness of the eye, and does not depend altogether on the instrument. We will write it as F, remembering that its appropriate value for a given brightness of field may be obtained from the results of König and Brodhun. Then

$$\delta U = \frac{F}{4} \tan U,$$

or

$$\delta U = \frac{F}{8} U' \text{ (very nearly, when U is small)},$$

where U′ is the angle between the planes of polarization of the two halves of the field.

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* In both the Laurent and Lippich half-shade systems the maximum brightness in the two sections of the field is slightly different owing to reflection and absorption in the plate and auxiliary prisms respectively. Hence the angle U will be slightly different in the two halves, but the theory given here remains substantially correct.

We must therefore arrange that the angle between the polarization planes of the two parts of the field is small in order that we may not have to go far from the extinction point to make the equality setting. Also, we must be careful to make the adjustment when both halves of the field are " equally dark " rather than when they are equal at the higher level of brightness when U is comparatively large. With $U = 2\frac{1}{2}°$ and $F = 0{\cdot}02$, $\delta U = 0{\cdot}012°$, but it is evident that, in order that the setting may be made satisfactorily, the light must have a fair intensity, otherwise the photometric sensibility of the eye may fall far below 2 per cent. As in photometry, the balance between instrumental and visual sensitiveness has to be adjusted. U must not be made too small. Another cause which operates to an increased extent with very small half-shadow angles is the residual elliptical polarization generally present to some extent in the light reaching the analyser. This is generally due to weak double refraction in the parts of the instrument between polarizer and analyser.

In most polarimeters nowadays the analysing angle may be varied at the will of the observer between 0° and 10° or so, and in practice the simple polarizer and analyser are never used. The settings in the best instruments for research work can be repeated to 0·001°, but in the ordinary commercial work results are required only to 1′ or perhaps 0·01°.

Transition Tint Instrument.—An end-point device which allows of the use of white light was devised by Soleil and others in the " transition tint " quartz plates. Each half of the field of view is filled by plates of right-handed and left-handed quartz respectively. The thickness of each (3·75 mm.) was just sufficient to rotate the " mean yellow " light through 90° in opposite directions, when it could be simultaneously extinguished by the analyser on each side of the field, leaving the other wave-lengths in less or greater proportions. The two halves of the field appear coloured, and any motion of the analyser results in a relative change of tint. Exact equality of tint is the criterion of the setting.

The distinction between the results obtained by the old transition tint polariscopes and the more modern instruments should be noted, as the particular wave-length for which the polariscope worked is not the D sodium light, but the mean yellow:

$$\frac{\text{D light rotation}}{\text{Mean yellow rotation}} = \frac{21{\cdot}7}{24{\cdot}0}.$$

Such a transition tint device is used in the Soleil-Ventzke-Scheibler transition tint saccharimeter constructed by Messrs. Schmidt & Haensch, but the instrument is not as widely used as the half-shade type, and has been practically abandoned in modern times.

Lummer's "Contrast Type" End Point.—Lummer devised an end-point device employing four prisms. The setting depended on the contrast principle described on p. 215, but it did not come into general use.* Wild's polaristrobometer† is another instrument not now frequently met with.

General Features.—In dealing with the main optical features which are perhaps peculiar to the polariscope, it may be pointed out that many of the precautions which have been discussed in connection with photometry apply in this case also. Thus Glan-Thompson prisms with normal end faces are employed in modern instruments. The remarks as to field brightness, angular subtense of field, &c., hold with very little reservation.

Wedge Compensators.—An important device due to Soleil, and first introduced into an instrument in 1848, is the use of quartz wedges for replacing in saccharimeters the divided circle commonly employed in the polarimeter.

If an accurately formed quartz wedge, in which the path of the light is parallel to the optic axis and perpendicular to the length of the wedge, be introduced into the beam, the rotation produced by an optically active solution may be annulled by that of the quartz, and if the surfaces of the wedge are accurately plane, the rotation produced, being proportional to the thickness of the crystal traversed, will therefore bear a linear relation to the displacement of the wedge.

Various arrangements are employed for this action, the commonest being to provide one fixed wedge (B in fig. 165 (*a*)) and one movable wedge (A); both are left-handed. The opposed angles of the two wedges prevent dispersion or deviation of the beam. If an extra plate C of right-handed quartz is provided, the rotation may be annulled while A is in the centre or any other part of its slide. One movement of A (upwards in the diagram) would introduce right-handed rotation, the opposite movement introduces left-handed rotation. Naturally the scale has to be calibrated. It is usually arranged so that the vernier on the movable wedge shall read zero when the lengths of the wedge on each side of the axis of the instrument are about 1 to 4 (to apex and base of the prism respectively). This allows of negative readings on the sugar scale, which is a useful provision when Clerget's method of " inversion " is employed. This consists in changing the pure cane-sugar into " invert " sugar of opposite rotation. By this means the proportion of sugar can be determined even in the presence of other optically active substances. For details see Bureau of Standards Circular No. 44 (Washington).

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* *Zeit. f. Inst.*, **16**. 209 (1896).

† Wild, *Ueber ein neues Polaristrobometer*, Berne, 1865.

In some cases two movable wedges are provided (fig. 165 (*c*)), and are moved separately, when they are known as the " control " and " working " wedges respectively. The control wedge, which is only moved by a special key, serves to adjust the zero for a suitable position of the working wedge. Ordinary precaution suggests the wisdom of moving only one part at a time if accurate results are desired. The linear displacements are indicated by a scale and vernier which is generally observed by means of an auxiliary eyepiece just above or beside the eye point of the main instrument. As the rotary dispersions of quartz and sugar solutions are about equal, the wedge system can be used as a compensator with white light. It is better, however, to use a light filter, such as a potassium bichromate solution or one of the standard yellow filters supplied by makers for the purpose.

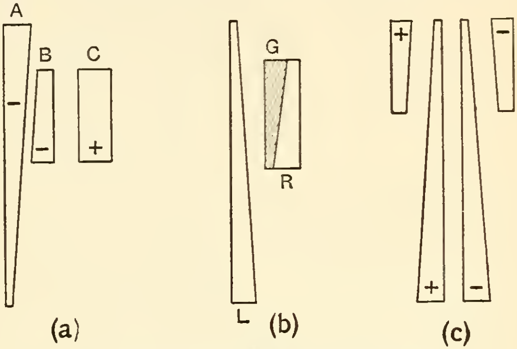

Fig. 165.—Quartz Wedge Compensators
(*a*) Early type (solid). (*b*) Marten's modified type.
(*c*) Double wedge.

Martens* has described an alternative construction of the wedge system which is illustrated in fig. 165 (*b*). In this arrangement left- and right-handed prisms L and R of quartz are employed with their thin ends similarly directed. L is the long movable wedge and R is fixed. The deviation which would be produced by the pair is annulled by the glass wedge G, which is oppositely placed and is cemented to R in order to save internal reflections.

Bellingham and Stanley doubt the wisdom of employing long wedges of quartz. They point out that the crystal is liable to twinning, and that stringent tests will reveal imperfections in the majority of quartz plates intended for optical working. It may be calculated that for the average accuracy required in compensator wedges the surfaces must be true planes to within one wave-length of sodium light.

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* *Zeit. f. Inst.*, 20, p. 82.

While it is true that this condition is one well within the powers of modern optical manufacturing processes, yet it is one which allows of no margin for accidents. Quartz wedges are usually finished off separately by hand, and the surfaces should be tested with great care.

In order to obviate the necessity of employing wedges as long as 5 cm. or so, Bellingham and Stanley have devised a construction in which the movable wedge is reduced in length to 2 cm., thus allowing of the selection of specially homogeneous portions of quartz for construction. In the complete compensator system two wedges are employed, as in previous cases, but each wedge consists of a pair of opposed wedges of right-handed and left-handed quartz. Each member is cut to a circular shape like a prismatic spectacle glass and the parts are placed in contact to form a parallel plate. The construction will be understood by reference to the sketch (fig. 166). It is evident that, in terms of optical rotation, the power of the "wedge" thus formed will be greatest along the direction OA, and least, in fact zero, along the diameter OB, where the relative thickness of right-handed and left-handed quartz does not change. If now such a wedge can be rotated while it is mounted in its slide, the effective angle, considered parallel to the slide, of the wedge is altered and the actual rotation, indicated by the scale on the moving part, can be made to approximate as closely to the truth as the most exact conditions of observation will allow. Two wedges of this construction being employed, the smaller wedge can be rotated eccentrically for the adjustment of the zero of the scale, while the larger movable one can be rotated to adjust the measure of rotation corresponding to a given displacement. To compensate for the shorter length the moving part carries a glass scale which can be read with accuracy by a suitable microscope. The compensator is illustrated in fig. 167. Set screws 2′ and 3′ are employed to rotate the wedges. The glass scale is shown by 7, the mounting and

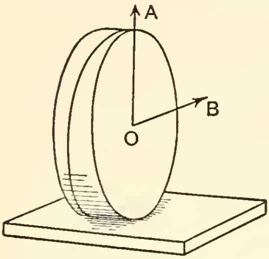

Fig. 166.—Pair of Wedges cut to Circular Shape

rotation scale for the movable wedge by 4, and the fixed wedge by 6.

Construction of a Polarimeter.—Fig. 168 represents diagrammatically the arrangements of a modern type of polarimeter.

N_1 and N_2 are the polarizer and analyser respectively. E is the end-point device, in this case consisting of the Lippich arrangement of two small auxiliary prisms.

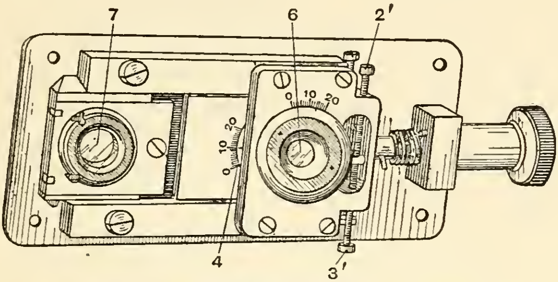

Fig. 167.—Bellingham and Stanley's Compensator

In order to secure that the dividing edges of the field shall appear quite sharp, the two inner faces of the auxiliary prisms may be slightly inclined (p. 241), being nearer together towards the letter *e* in the diagram. As the condenser lens C has, generally speaking, to be used with a fairly close source of light, it is made of fairly high power, preferably so that the image of the source is brought to a focus in the region of the object glass of the observing telescope.

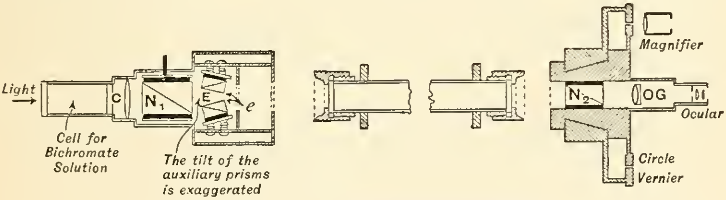
Light
C
N₁
E
e
Cell for Bichromate Solution
The tilt of the auxiliary prisms is exaggerated
Magnifier
N₂
OG
Ocular
Circle
Vernier

Fig. 168.—Diagram of Parts of Lippich Polarimeter

The end point device thus appears filled with light of a fairly uniform brightness.

Users should beware of employing any auxiliary condenser to focus the image of the source on the end point E, as the variations in intensity found over a sodium flame or mercury lamp would destroy the value of the settings.*

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* Landolt (*Optical Rotation of Organic Substances*, 2nd edition, 1897) discusses the illumination at greater length and shows that, where the end-point device has to be employed with parallel light, it becomes necessary to restrict the area of the illuminating source.

A front tube is provided which can be filled with a solution for a light filter. The stray light is thus removed, such as the faint blue luminosity of the Bunsen flame, if such is employed.

The instrument in the figure is provided with a divided circle. Verniers and circle are in the same plane, and readings are taken by means of a magnifier with a telecentric stop. The telescope at the end is of the inverting type, although some instruments are provided with a Galilean telescope. In the present case the image of the end-point device is made to fill the circular aperture in the telescope diaphragm, thus securing a neater appearance of the field. During observations the pupil of the eye must be brought into coincidence with the exit pupil of the telescope, or rather the image of the analyser diaphragm formed by the lenses of the telescope system. In such instruments as these the exact method of construction of the mechanical parts is not of superlative moment, the optical parts being of much greater importance. The principal moving part is the divided circle.

The mounting of the auxiliary prisms at E is, however, a matter of some delicacy. Each is cemented to a metal mount, the position of which is controlled by adjusting screws. These screws are protected by a metal cover which can only be removed with some difficulty (owing, of course, to the necessity of keeping this delicate adjustment from the unthinking interference of careless hands), but the relative angular position of the polarizer N_1 may be readily adjusted, and a fine adjustment key, not shown in the sketch, is provided for adjusting the position of N_2 relatively to its scale.

The tubes to contain solutions are closed at each end by plane glass end pieces, lightly held against the ground ends of the tube by the screwing up of the metallic end pieces. Rubber bands are sometimes employed, but in no case should the end pieces be subjected to any strain capable of producing polarization effects. The end pieces must be chosen from glass which shows no sign of strain due to bad annealing. The clear aperture of the tubes in this instrument is 9 mm., and the length is 20 cm. For work with ultra-violet light these end pieces may be made from double quartz plates. The tubes are placed in a trough with a hinged cover. In the better instruments the trough is adjustable, by means of screw movements, relatively to the optical parts so as to allow of the use of differing varieties of tube, and further, the cover is made so that it revolves about the axis of the system. This obviates the use of hinges which, especially in saccharimeters, are apt to be corroded by contamination with liquid.

The tubes in modern instruments are often made of brass, and are 9 mm. in diameter. Care is necessary in construction to ensure that the ends are perpendicular to the axis. The caps holding the end windows should not support the weight of the tube, which should be carried by rings on the body as shown in fig. 168. In tubes due to Bellingham and Stanley the metal ends can be fixed to the tube by a packing gland. In some cases a recess may be formed at one end of a tube to hold the small bubble of air which is sometimes accidentally formed when the end pieces are put on.

Saccharimeters are polarimeters carrying a special scale to indicate sugar values, the scale being derived in the following way. Granted that the same "length of solution" is always employed, the 100 division on the scale always corresponds to the setting for a definite weight of pure cane-sugar dissolved in 100 c.c. of solution. The temperature is to be kept at 20°. If then the same weight of an unknown substance containing sugar is dissolved in the same volume of solution, the reading in the saccharimeter scale directly expresses the percentage of cane-sugar in the mixture, leaving out of account the various small errors which have been mentioned. The "normal" weight of the standard commercial saccharimeter is 26·0 gm., and the range 0 to 100 on the scale corresponds to about 34·62°.

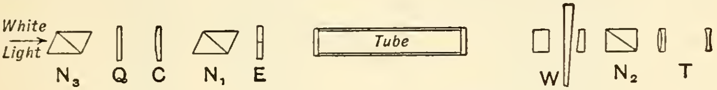

Fig. 169.—Optical Train of Soleil-Ventzke-Scheibler Saccharimeter

The Soleil-Ventzke-Scheibler saccharimeter, although now little used, still merits a short description. As was previously indicated, it makes use of a transition tint end-point device.

The diagram (fig. 169) shows the optical system. N_1 and N_2 are the polarizer and analyser respectively. E is the transition tint quartz plate. W is the Soleil wedge system, and T is the telescope, in this case of the Galilean type.

Disregarding the parts N_3 and Q for the moment, it will be seen that the first necessity is to adjust the relative positions of N_1, E, and N_2 until the yellow rays are extinguished in each half of the field. When this is the case, the tint will be a uniform rose-violet, but contrasting colours (more red and blue respectively) are seen immediately the Nicol is moved out of its correct position. It is necessary to take out the quartz wedge compensator system altogether as a preliminary to the adjustment of N_2. The relative position of these parts is never altered in ordinary work; any rotation introduced by a tube of sugar or other solution can be compensated by the quartz wedge system.

Q and N_3 are a quartz plate and Nicol respectively; by rotation of these parts relatively to N_1 it is possible to vary the tint at which equality is reached in the two halves of the field. Although the colour produced by the system Q and N_3 persists through the instrument, it is clear that since the polarizer N_1 *can emit nothing but light polarized in one direction*, although coloured perhaps, the action of the remainder of the optical train is not prejudiced. Different observers choose various "sensitive tints" and modify the rose-violet generally to something nearer a grey, but the action of any rotation of the plane of polarization produced by an optically active substance is always to introduce a red-blue contrast.

The sensitive tint produced is especially useful in case coloured solutions have to be dealt with, as any strong coloration may sometimes be sufficiently counteracted to enable the sensitive tint method to be used.

In other ordinary details the instrument differs little from the half-shade type of saccharimeter.

General Remarks.—The majority of saccharimeters necessarily agree with the above example in having a fixed half-shadow angle. In the Bates saccharimeter* the half-shadow angle can be changed at will, the necessary readjustment of the analyser being effected automatically. The accuracy of the compensation has been a matter of some considerable discussion.

From the maker's point of view, the important details in the construction of a saccharimeter are the accurate formation of the optical parts, including the strict parallelism of the faces of a rotating Nicol (which prevents an unsatisfactory wobble of the field of view). If ordinary Nicols are employed a compensating prism of glass is necessary.

The saccharimeter is frequently employed in tropical climates, especially on sugar-growing plantations, and being an industrial instrument it is often subjected to somewhat rough usage. With this condition in mind, several desirable features for the construction of the instrument may be cited.

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* *Bull. Bur. Standards*, 1907, Vol. IV, p. 461; 1908, Vol. V, p. 193.

1. It should be mechanically robust, and all except essential joints and movements should be eliminated.

2. The stand, supports, and housing of the optical parts should be simple and easy to clean. It is an advantage if cleaning may be done by sponging with water.

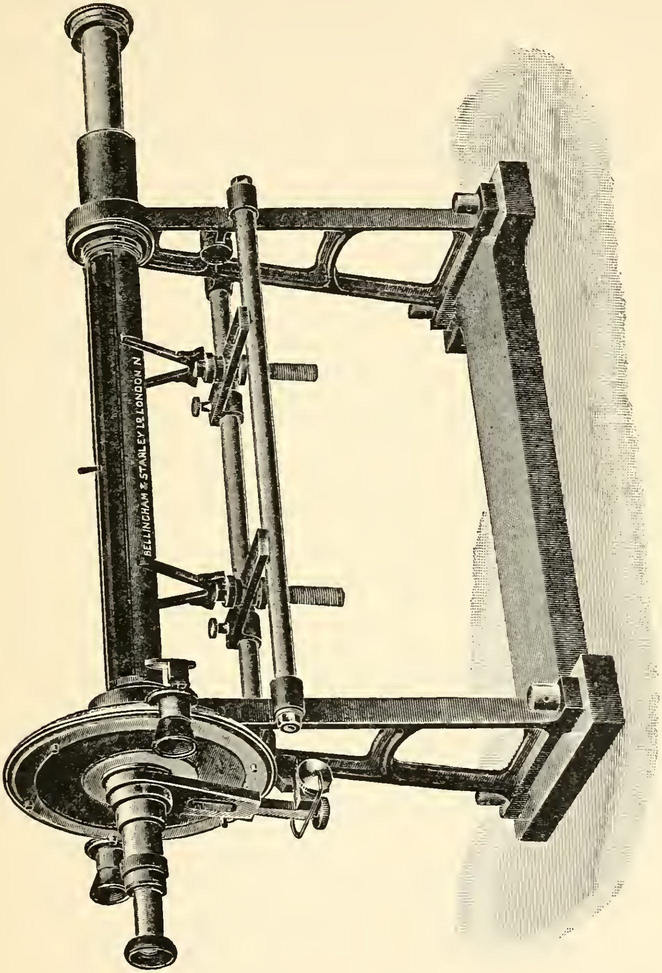
BELLINGHAM & STANLEY LD LONDON N

Fig. 170. Bellingham and Stanley's Polarimeter

3. Sensitive optical parts like the end-point device could, with advantage, be protected by renewable windows of unstrained glass.

A Modern Polarimeter.—Fig. 170 illustrates a typical form of polarimeter due to Messrs. Bellingham & Stanley.

The disc carrying the divided circle is bevelled off at 45° on the side away from the eyepiece, and the graduations are made on the bevelled face, the vernier forming segments of the same conical surface which contains the divided ring. They are read by means of reflectors seen in the figure on each side of the eyepiece. The instrument contains the special half-shade device previously described.

The general construction is sufficiently massive to avoid shake when the moving parts are handled.

CHAPTER XIV

Errors and Accuracy of Observations

Treatment of Results.—In the practical use of instruments it is soon found that accurate results—that is, results as accurate as can be yielded by a method under the best conditions—are by no means easy to obtain. Therefore it is very desirable that the user of a measuring instrument shall be able both to use his instrument to the best advantage and treat his observations to the best advantage also. He should also be able to form an estimate of the accuracy of his work and final results.

Let us consider the nature of errors likely to occur in observational work. We may group them as follows:

1. Constant or regular errors, amongst which we find:

(*a*) " Instrumental errors ", such as errors of graduation, periodic errors of screws, &c. These must be removed, or allowed for if necessary, after careful investigation and measurement.

(*b*) " Personal errors ", depending on the judgment of the observer. A regular error of this type (" personal equation ") may often be eliminated from a final result by such devices as reversing the order or direction of observations over part of the work.

2. Irregular or accidental errors.

These are errors depending on unavoidable temporary changes outside the control of the observer, such as effects due to vibration of the building, temperature changes, &c.

The distinction between regular and irregular errors is not easily observed in all cases. By exercising great caution more and more conditions may be brought under control, and their effects allowed for. The remaining errors will be due to causes of which the observer has no knowledge, or which he has no opportunity to eliminate.

In making measurements the investigation of the sources of error must be carried out systematically, and their probable influence *on the final result* should be estimated. Thus, although it would be useless to apply a temperature correction of 0·5 per cent, say, to a result found by a single observation with a probable error of 5 per cent, yet the same temperature correction would become of much greater relative importance if by the mean of many observations the probable error of the final mean result was reduced below 1 per cent.

Experimental details, then, are to be arranged so as to eliminate, automatically if possible, any regular errors. As an example of this the avoidance of errors due to the eccentricity of a circle by taking opposite vernier readings may be instanced.

Finally the results of observation are to be combined so that the outstanding irregular errors may have the least probable effect on the results, regular and known errors being allowed for.

A number of observations of some quantity which have been made with the greatest care under similar circumstances should possess equal weight in determining the true result, or its most probable value. Each result may be regarded as an equal indication of the truth.

Other conditions may give greater weight to observations. We should be more willing to trust determinations when employing a delicately poised instrument such as a balance in still air than we should if there were uncertain breezes. We seek to determine the most probable correction to be applied to separate observations by a consideration of the observations themselves.

The Arithmetic Mean.—Suppose we have the measurements (a and b) of some quantity. They have equal " weight ", and consequently there is no reason for supposing the necessary correction to one to be greater than that to the other. Hence the *most probable* value of the final result is $\frac{a+b}{2}$. The extension of a slightly more complicated argument to more than two observations shows that the arithmetic mean of a number of observations gives the most probable value of the quantity, each observation having an equal influence on the result. An important characteristic of the arithmetic mean is that, if we compute the residual differences between the values of each observation and the mean, we find that the sum of the squares of such residuals is a minimum.

Taking Observations.—A concrete example of the method of taking observations will be useful.

Suppose we have a regular pattern of somewhat ill-defined interference fringes, and we wish to determine, by a cross wire moving with a scale and vernier, the distance between each fringe. Call the fringes 0, 1, 2, 3, 4, 5, 6, &c. We could proceed by setting on fringe 0 and fringe 10, taking the difference of readings and dividing by 10. It is clear that no advantage would be gained by setting on each intermediate fringe, finding the indicated distance between successive fringes, and then taking the mean. Such a procedure is merely a waste of time.

It is much better, if we have 15 fringes, say, to make settings of the cross wire on fringes 1, 2, 3, 4, and 5, also on 11, 12, 13, 14, and 15, thus finding the intervals 1 to 11, 2 to 12, &c. The mean of these 5 intervals divided by 10 will give us a much better result for the distance between the fringes.

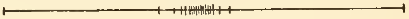

Fig. 171

Probable Error.—In determining a quantity we shall in general make certain errors.

If, for example, 100 people independently attempted to bisect a line, we might by many trials obtain an appearance of the kind suggested in fig. 171.

The smaller errors will be the most numerous, greater ones the most rare. The relative frequency of occurrence of an error within small limits on each side of a certain value is known as the "probability". Thus if an error greater than Δ occurs n times in m observations, the probability of an error greater than Δ is $\frac{n}{m}$, where n and m are supposed to be large numbers. Naturally we should never attempt to judge the likelihood of error from a few results only. The probability (y) that an error may have a value not greater than Δ depends on the magnitude of Δ. This is expressed mathematically by

$$y = \phi(\Delta).$$

We noticed above that the larger errors are likely to be the less frequent, and we shall generally find the probability of small positive and negative errors of the same numerical value about equal.

Consider the expression

$$\delta y = ce^{-(h^2\Delta^2)}\,\delta\Delta,$$

where c and h may be considered constants and e is the base of natural logarithms. δy has a maximum value when $\Delta = 0$, and it approaches zero when Δ is large. Also y is independent of the sign of Δ. The general shape of the curve representing this expression is shown in fig. 172, where δy is the ordinate and Δ is plotted as the abscissa. As a matter of fact, this is the type of expression for the " error function " which is found by logical deduction from our definitions of probability; δy represents the chance that an error will fall between the close limits Δ and $\Delta + \delta\Delta$. Fully stated the expression is

$$\delta y = \frac{h}{\sqrt{\pi}} e^{-(h^2\Delta^2)} \delta\Delta,$$

where h depends on the *measure of precision* of the observations. For the theoretical derivation of the above expression reference may be made to textbooks on " least square " methods. Treating δy and $\delta\Delta$ as differentials, we will integrate in order to find the probability (y_1) that the error lies between the limits 0 and a. Then

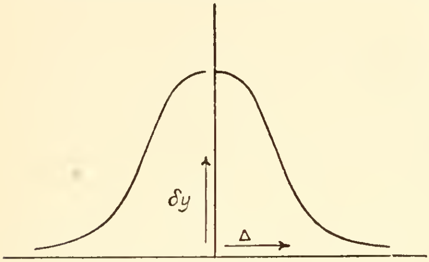

Fig. 172.—The Error Curve

$$y_1 = \frac{h}{\sqrt{\pi}} \int_{\Delta=0}^{\Delta=a} e^{-(h^2\Delta^2)} d\Delta.$$

Putting $t = \Delta h$ the integral becomes

$$= \frac{1}{\sqrt{\pi}} \int_{t=0}^{t=ah} e^{-t^2} dt.$$

The probability that the error will lie between $+a$ and $-a$ is thus given by

$$\frac{2}{\sqrt{\pi}} \int_{t=0}^{t=ah} e^{-t^2} dt.$$

Note that the probability will be a fraction representing the value $\frac{n}{m}$ where we find an error between these limits n times in m observations.

By evaluating the integral for different values of t we may find at once the fraction representing the probability of errors of corresponding magnitudes. For example, the probability that an error would lie between the limits represented by $t = 0$ and $t = 0{\cdot}5$ is found to be 0·52. Hence 520 errors per thousand would lie between the limits indicated. The limits are controlled, of course, both by the magnitude of Δ and the measure of precision h. Evidently a large h corresponds to small average errors. The shape of the curve representing the expression will be controlled by the value of h, a large h giving a sharper peak. The following table gives the results of integration.

Limits of t.		Numbers of Errors per 1000 Observations.
0 to 0·5	....	520
0·5 „ 1·0	....	322
1·0 „ 1·5	....	123
1·5 „ 2·0	....	29
2·0	....	5

In comparing the errors of two series of observations we may select *the error which occupies the middle place* in each series of errors when they are arranged according to their magnitude (all being considered of the same sign), so that the number of errors which are greater than this one equals the number of errors which are less. Evidently the error r, satisfying this middle place condition, is that one for which the value of the integral is 0·5.

Denoting the corresponding value of t by p, we find by interpolation from the table

$$p = 0{\cdot}477, \quad \text{when} \quad \frac{2}{\sqrt{\pi}}\int_0^p e^{-t^2}dt = \tfrac{1}{2}.$$

To put the equation in words: there are even chances that the error will be less than r. Hence r is called the probable error. The measure of precision $h = \dfrac{p}{r}$.

The Mean Error.—Another quantity must now be defined. This is called the " mean error ", and is the error the square of which is the mean of the squares of all the errors. We must again take a result which can be mathematically deduced,

$$r = 0{\cdot}674 \times \mathrm{E},$$

where E is the mean error as defined above.

Although these factors are the true results following from our definitions and ideas of " probability ", it is evident that their use simply depends on these ideas alone, and there is nothing *necessarily* fundamental about them. It would be pedantic to use the factors to the last decimal, but they give a useful convention by means of which the relative precision of results may be indicated. By actual experiment, it has been proved that the probability of errors is represented to a fair degree of accuracy by the expressions developed theoretically.

The Mean Error, " E ".—In considering a series of observations, let n, n', n'', &c., be the observed values of a quantity. Let $x_0 \equiv$ the arithmetic mean, and let ν, ν', ν'', &c., be the residuals defined by the equations

$$\begin{aligned} \nu &= n - x_0, \\ \nu' &= n' - x_0, \\ \nu'' &= n'' - x_0, \text{ \&c.} \end{aligned}$$

If x_0 were *certainly* the true value of x, then ν, ν', &c., would be the true errors and consequently we should have

$$m\mathrm{E}^2 = \nu^2 + \nu'^2 + \nu''^2 + \text{\&c. to } m \text{ terms},$$

which series may be written in the usual notation for least square methods as

$$m\mathrm{E}^2 = [\nu\nu].$$

Hence the equation

$$\mathrm{E} = \sqrt{\frac{[\nu\nu]}{m}}$$

gives a close approximation for a long series of observations, although we can see that the value of E may be slightly smaller than it should be, because with a finite series of observations x_0 will not generally be coincident with the true value. It may be shown that a more truthful formula is

$$\mathrm{E} = \sqrt{\frac{[\nu\nu]}{(m-1)}},$$

whence $$r = 0{\cdot}674\sqrt{\frac{[\nu\nu]}{(m-1)}}.$$

A modification derived by Peters* gives a modified form:

$$r = 0{\cdot}84\frac{[\nu]}{\sqrt{m(m-1)}}.$$

A considerable number of results are needed in order to make these formulæ significant.

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* *Ast. Nach.*, Vol. XLIV, p. 33.

Mean and Probable Errors of Functions of Independent Observed Quantities.—In making a physical measurement we shall often derive the final result as the sum or product, &c., of two or more independently observed quantities. It is obviously very important to inquire how errors in these will affect the final result.

Taking the simplest possible case, let

$$X = x_1 + x_2,$$

where a series of observations for x_1 and x_2 are made, giving several values for x_1, all with certain amounts of error, Δ_1, Δ_1', Δ_1'', &c., while the corresponding values for x_2 have errors Δ_2, Δ_2', Δ_2'', &c. The consequent errors of X are given by

$$\Delta_1 \pm \Delta_2,\ \ \Delta_1' \pm \Delta_2',\ \ \Delta_1'' \pm \Delta_2'', \text{ \&c.}$$

Denoting the mean error of X by E, we have by definition

$$\begin{aligned} mE^2 &= (\Delta_1 \pm \Delta_2)^2 + (\Delta_1' \pm \Delta_2')^2 + (\Delta_1'' \pm \Delta_2'')^2 + \text{\&c.} \\ &= [\Delta_1\Delta_1] \pm 2[\Delta_1\Delta_2] + [\Delta_2\Delta_2]. \end{aligned}$$

Since in a great many observations there will probably be as many positive as negative products of the form $\Delta_1\Delta_2$, we shall have

$$[\Delta_1\Delta_2] = 0;$$

and if we define $mE_1^2 = [\Delta_1\Delta_1]$, and $mE_2^2 = [\Delta_2\Delta_2]$, then

$$E^2 = E_1^2 + E_2^2.$$

This can be extended to any number of symbols; when we have

$$X = x_1 \pm x_2 \pm x_3 + \text{\&c.},$$

then we shall find

$$E^2 = E_1^2 + E_2^2 + E_3^2 + \text{\&c.}$$

Also if

$$X = a_1x_1 \pm a_2x_2 \pm a_3x_3 + \text{\&c.},$$

we shall have

$$E^2 = a_1^2E_1^2 + a_2^2E_2^2 + a_3^2E_3^2 + \text{\&c.}$$

In consequence of the constant ratio between "mean" and "probable" errors the foregoing results can be applied to probable errors also.

Effect of Increasing the Number of Observations.—We can now deduce a result of great importance, as it is extremely necessary not only to know the probable error of a single observation, but also to be able to estimate the probable error of a *mean result.*

Let x, x_1, x_2, &c., be several observed values of the same quantity, their arithmetic mean (x_0) being

$$x_0 = \frac{1}{m}(x + x_1 + x_2 + \&c.).$$

Let r be the probable error of each observation, all observations having the same weight. Then by the foregoing results the probable error of the *sum* is equal to

$$\surd(r^2 + r^2 + r^2 + \ldots \text{ to } m \text{ terms}) = \surd(mr^2) = r\surd m.$$

The probable error of $\frac{1}{m}$th of the sum $= r_0 = \frac{r\surd m}{m} = \frac{r}{\surd m}$.

The probable error of the arithmetic mean will thus be found by the formula

$$r_0 = 0{\cdot}674\sqrt{\frac{[vv]}{m(m-1)}},$$

and we have also

$$E_0 = \sqrt{\frac{[vv]}{m(m-1)}}.$$

General Case of a Function of Several Independent Observed Quantities.—Returning to the general investigation, we must investigate the case in which X is any function whatever of the independent observed quantities.

$$X = f(x, x_1, x_2, \&c.).$$

Let us consider that any one of the above quantities may be expressed,

$$x_\mu = a_\mu + x_\mu',$$

where a_μ is a part not subject to error and x_μ' is the part subject to error. In the usual case x_μ' is a small quantity of the first order, the square of which may be disregarded.

The function X developed by Taylor's theorem will approximate to

$$X = f(a, a_1, a_2, \&c.) + \left(\frac{dX}{dx}\cdot x' + \frac{dX}{dx_1}\cdot x_1' + \frac{dX}{dx_2}\cdot x_2' + \&c.\right)$$

+ higher order terms which will be negligible when x_μ' is small.

The probable error of X is now the same as the probable error of the second term in the Taylor expansion, i.e. of

$$\frac{dX}{dx}\cdot x' + \frac{dX}{dx_1}\cdot x_1' + \frac{dX}{dx_2}\cdot x_2' + \&c.,$$

and we can thus put, by a previous result, remembering the proportionality of mean and probable errors,

$$R^2 = \left(\frac{dX}{dx}\right)^2\cdot r^2 + \left(\frac{dX}{dx_1}\right)^2 r_1^2 + \left(\frac{dX}{dx_2}\right)^2 r_2^2 + \&c.,$$

where R is the probable error of the derived quantity X.

Possible Error.—In a series of observations the greatest deviation from the mean result is sometimes taken as an indication of the extent of the *possible* error and serves as an indication of the general precision.

Weight of Observations.—Measurements or observations made under essentially similar conditions with the same apparatus by one observer on the same occasion will probably all have an equal claim to be considered as representing the truth. Suppose that an observer A makes a series of x measurements on a certain quantity and obtains a result with a calculated mean error e_1, say. The error of the arithmetic mean $e_0 = \frac{e_1}{\sqrt{x}}$. Another observer B after taking the same number of observations (x) finds his mean error ne_1, say. If he could go on and take n^2x observations of about the same accuracy he would reduce the error of his arithmetic mean to $e_0' = \frac{ne_1}{\sqrt{n^2x}} = \frac{e_1}{\sqrt{x}}$. This is expressed by saying that n^2 observations from B have the same weight as one observation from A.

If we have several measurements a, b, c, d of a certain quantity of respective weights l, m, n, &c., the most probable value of the final mean is found by imagining that we have l observations giving the result a, m observations giving the result b, and so on, finding at last the arithmetic mean of these (l, m, n, &c.) fictitious observations.

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